Derived categories and scalar extensions

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Pawel Sosna
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1. Gutachter: Prof. Dr. Daniel Huybrechts
2. Gutachter: Prof. Dr. Jan Schröer

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Summary

This thesis consists of three parts all of which deal with questions related to scalar extensions and derived categories.

In the first part we consider the question whether the conjugation of a complex projective K3 surface X by an automorphism of the complex numbers can produce a non-isomorphic Fourier–Mukai partner of X. The answer is affirmative. The conjugate surface is thus in particular a moduli space of locally free sheaves on X. The proof consists of constructing non-isomorphic conjugate derived equivalent K3 surfaces over an extension field of $\overline{\mathbb{Q}}$ and then lifting the situation to the complex numbers. We use our result to give higher-dimensional examples of derived equivalent conjugate varieties. We furthermore prove that a similar result holds for abelian surfaces.

The topic of the second part is the behaviour of stability conditions under scalar extensions. Namely, given a variety X over some field K and its bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$, one can associate to it a complex manifold of stability conditions, denoted by $\mathrm{Stab}(X)$. We compare the manifolds $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X_L)$ for a field extension L/K. For the most part we only consider the case of a finite Galois extension. In particular, we prove that in this case $\mathrm{Stab}(X)$ embeds into $\mathrm{Stab}(X_L)$ as a closed submanifold. Since the topology on the stability manifold is closely related to the numerical Grothendieck group of $\mathrm{D}^{\mathrm{b}}(X)$ we also study the question whether the stability manifold can change under scalar extension if the numerical Grothendieck group does not. The answer is that $\mathrm{Stab}(X_L)$ could only acquire new connected components. This result is applied to the stability manifold of a complex K3 surface.

In the third and last part we consider the following question: Can one naturally define an L-linear triangulated category $\mathcal{T}_L$ if a K-linear triangulated category $\mathcal{T}$ and a field extension L/K are given? Our guiding example is the passage from $\mathrm{D}^{\mathrm{b}}(X)$ to $\mathrm{D}^{\mathrm{b}}(X_L)$. We propose a construction and prove that our definition gives the expected result in the geometric case. It also gives the anticipated result when applied to the derived category of an abelian category with enough injectives and with generators. We furthermore prove that in the just mentioned cases the dimension of the triangulated category in question does not change for finite Galois extensions.

Contents

Introduction

Classical algebraic geometry deals with varieties over some fixed algebraically closed field, usually the field of complex numbers. This point of view is the source for geometric intuition, but one of the fundamental advantages of the language of schemes is that it allows us to do algebraic geometry over arbitrary fields and, more generally, over commutative rings. This approach is much more flexible, but probably sometimes not as intuitive.

One of the fundamental constructions in algebraic geometry is the fibre product of geometric objects. A special case of this construction is the notion of *scalar extension* or *base change*: Given a scheme X over some field K and a field extension L/K, one can consider the *base change scheme* $X_L = X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(L)$. This is an example of the general philosophy emphasised by Grothendieck that instead of working with a variety over some fixed field and study its properties, one should instead study the properties of the morphism $f : X \longrightarrow \mathrm{Spec}(K)$. Clearly, from this point of view it is important to understand the behaviour of properties of f under scalar extension. For example, if $X \longrightarrow \mathrm{Spec}(K)$ is smooth, then also $X_L \longrightarrow \mathrm{Spec}(L)$ is smooth: One says that smoothness is stable under base change. On the other hand certain geometric concepts do not have this property: For instance, X might be a connected space, whereas X_L is not.

In his paper [44] Mukai proved that the Poincaré bundle on $A \times \widehat{A}$, where A is a complex abelian variety and $\widehat{A}$ its dual abelian variety, defines an equivalence between the derived categories of coherent sheaves on A and $\widehat{A}$. Since the two varieties in this example are in general not isomorphic, it became clear that the derived category provides a new geometrical invariant. Generalising this special case one calls two varieties X and Y over some field K *Fourier–Mukai partners* (FMP) or *derived equivalent* if there is an exact K-linear equivalence between the derived categories of coherent sheaves on X resp. Y. Usually only smooth projective varieties are considered since in this case the derived category is reasonably big and by a theorem of Orlov [53] every equivalence is induced by an object on the product.

The purpose of this thesis is to study base change techniques in the context of derived, or more generally triangulated, categories. Thus, we study the effect scalar extensions can have on the derived category resp. on the geometry encoded by it.

Since Mukai's paper quoted above one large area of research is the investigation of possible Fourier–Mukai partners of a given smooth projective variety (in the literature the results are usually formulated over $\mathbb{C}$ but the arguments often work over an arbitrary field). Let us review some of the results. The easiest example is the case of a curve. It turns out that two curves are derived equivalent if and only if they are isomorphic, so the derived category already determines the variety. Another instance where this behaviour appears was investigated in [17]: The authors in particular prove that a surface of general type does not have any non-trivial Fourier–Mukai partners. A very prominent case was studied in [9]: If the canonical or the anti-canonical bundle of a variety X is ample, then any Fourier–Mukai partner of X is already isomorphic to X. Furthermore, it is possible to compute the group of autoequivalences of the derived category of X.

The last quoted results suggest that the derived category of a smooth projective variety X with trivial canonical bundle should be a particularly interesting object. It is known that any such X over the complex numbers is, up to a finite unramified covering, isomorphic to a product of abelian, Calabi-Yau and irreducible holomorphic symplectic varieties. Thus, it is natural to investigate the derived categories of the varieties of the three mentioned types. In dimension one one only has elliptic curves, which are both abelian and Calabi-Yau, and they do not have any non-trivial Fourier–Mukai partners. Orlov further investigated the case of abelian varieties in [54] and gave conditions for two abelian varieties to be derived equivalent. In particular, there are only finitely many non-isomorphic FMPs of a given abelian variety A. The next interesting case are K3 surfaces, which are the irreducible holomorphic symplectic varieties of the smallest possible dimension.

Due to results by Mukai [45] and Orlov [53] we have geometric and cohomological criteria for K3 surfaces to be derived equivalent. In particular, it follows that any given K3 surface has only finitely many non-isomorphic Fourier–Mukai partners. It is also possible to view derived equivalent K3 surfaces as elements in the orbit of the action of a certain discrete group on the moduli space of (generalised) K3 surfaces (see e.g. [32]). An account of some of the results can be found in Chapter 1.

There is a different group acting on the moduli space, namely the group of automorphisms of the complex numbers. Namely, given a complex projective K3 surface X and a $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have the notion of a conjugate K3 surface X^σ given by base change with respect to σ, see Definition 1.1.9. The K3 surfaces X and X^σ will in general be non-isomorphic as schemes over $\mathbb{C}$, but they are isomorphic over $K = \mathbb{Q}$ and thus there is a $\mathbb{Q}$-linear exact equivalence $\mathrm{D^b}(X) \simeq_{\mathbb{Q}} \mathrm{D^b}(X^\sigma)$. An obvious question is whether one can find examples where there is a $\mathbb{C}$-linear exact equivalence as well, without X and X^σ being isomorphic over $\mathbb{C}$. Thus, we would like to understand the connection between the actions of $\mathrm{Aut}(\mathbb{C})$ and the above mentioned discrete group on the moduli space of projective K3 surfaces. This is precisely the question which is investigated in Chapter 1. The first main result of this thesis is the following theorem which shows that the orbits of the two group actions can intersect in more than one point:

Theorem 1 (Theorem 1.3.6) *There exist a complex projective K3 surface X and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that the conjugate K3 surface X^σ is a non-isomorphic complex K3 surface, but there exists a $\mathbb{C}$-linear exact equivalence $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X^\sigma)$.*

The basic idea of the proof is to find a curve C in the moduli space of K3 surfaces over $\overline{\mathbb{Q}}$ which is invariant under a so called Mukai involution, the latter being a map sending a K3 surface X to a certain moduli space $M_X(v)$ of stable sheaves on X. The induced automorphism of the function field $K(C)$ will allow us to produce two non-isomorphic derived equivalent K3 surfaces over $\mathbb{C}$. Since this automorphism of $K(C)$ extends to an automorphism of $\mathbb{C}$ these two K3 surfaces will also be conjugate.

We use the above result to produce higher-dimensional examples. To be more precise we prove that there exist derived equivalent irreducible holomorphic symplectic varieties which are also conjugate (cf. Theorem 1.3.10). We furthermore show that a similar result holds for an abelian surface, namely that there exists a complex abelian surface A such that $\widehat{A}$ is a conjugate surface (cf. Theorem 1.4.1).

In Chapter 2 we turn our attention to stability conditions on triangulated categories. This concept was introduced by Bridgeland in [14]. One motivation was to understand a certain notion of stability in string theory. Another motivation is more general and was already alluded to previously: Try to extract geometry from homological algebra. Bridgeland proved that under mild assumptions the set of stability conditions forms a (possibly infinite-dimensional) complex manifold. If one considers so called numerical stability conditions on the derived category of a smooth projective variety X, then the stability manifold, denoted by $\mathrm{Stab}(X)$, is always finite-dimensional.

The stability manifold always lives over the complex numbers, even for the derived category of a smooth projective variety defined over a, say, finite field. Thus, considering a field extension L/K and a smooth projective variety X over K it is interesting to ask how the stability manifolds of X and its base change scheme X_L are related. The description of the topology on the manifolds (cf. Theorem 2.1.9) suggests that $\mathrm{Stab}(X)$ and $\mathrm{Stab}(X_L)$ might be different if their numerical Grothendieck groups are. On the other hand one might expect that if the numerical Grothendieck group does not change under scalar extension, then neither does the stability manifold. In order to tackle these questions we will, for the most part, assume that the field extension is finite and Galois.

Our second main result is the following

Theorem 2 (Theorem 2.2.22) *For any finite and separable field extension L/K the manifold $\mathrm{Stab}(X)$ is a closed submanifold of $\mathrm{Stab}(X_L)$.*

The proof consists of a detailed investigation of the maps between the stability manifolds induced by the functors p^* and p_* in the case where L/K is Galois. The more general case then follows easily. We use several results obtained in a slightly different context in [40]. One of the interesting facts in this situation is that the pushforward functor induces a continuous map $\mathrm{Stab}(X) \longrightarrow \mathrm{Stab}(X_L)$, whereas the pullback functor only defines a map $\mathrm{Stab}(X_L)_p \longrightarrow \mathrm{Stab}(X)$, where $\mathrm{Stab}(X_L)_p$ is a certain closed subset of $\mathrm{Stab}(X_L)$. We prove that in the Galois case the subset $\mathrm{Stab}(X)$ is a deformation retract of $\mathrm{Stab}(X_L)_p$. Thus, in a sense, the maps induced by the scalar extension only see a part of $\mathrm{Stab}(X_L)$ which is of the same homotopy type as $\mathrm{Stab}(X)$. It is not clear at the moment what the distinguished features of the complement of $\mathrm{Stab}(X_L)_p$ in $\mathrm{Stab}(X_L)$ are.

In the last section of Chapter 2 we investigate the behaviour of stability manifolds if one assumes that the numerical Grothendieck group does not change under scalar extension. The result is

Theorem 3 (Theorem 2.4.7) *Let L/K be a finite Galois extension. If the map*

$$N(X) \otimes \mathbb{C} \longrightarrow N(X_L) \otimes \mathbb{C}$$

induced by the pullback map is an isomorphism, $\mathrm{Stab}(X)$ *is non-empty and* $\mathrm{Stab}(X_L)$ *is connected, then we have a homeomorphism* $\mathrm{Stab}(X) \simeq \mathrm{Stab}(X_L)$.

Thus, in the case of a finite Galois extension and under the above assumption, the stability manifold can only acquire new connected components (note, however, that we do not have an explicit example where new components appear). One of the most interesting examples where one distinguished connected component of the stability manifold has been computed is the case of a complex K3 surface (see [15] and Example 2.1.13 for a brief account). Under certain assumptions on the K3 surface in question Theorem 3 applies and one can prove that the component is defined over the real numbers. For the precise statement see Proposition 2.4.8.

Let us further mention an auxiliary result obtained in Chapter 2 (Proposition 2.3.4): It gives a necessary and sufficient criterion for a t-structure on $\mathrm{D^b}(X_L)$ to descend to a t-structure on $\mathrm{D^b}(X)$, where L/K is a finite Galois extension.

The goal of Chapter 3 is to introduce scalar extensions for triangulated categories. Namely, to any field extension L/K and a K-linear triangulated category $\mathcal{T}$ we would like to associate an L-linear triangulated category $\mathcal{T}_L$. One possible motivation for this (apart from it being a very natural question) is the following: If L/K is a finite Galois extension with Galois group G, then one could define the category of G-linearised objects in $\mathrm{D^b}(X_L)$ as $\mathrm{D^b}(\mathrm{Coh}^G(X_L))$, the bounded derived category of the abelian category of G-linearised coherent sheaves on X_L. By Galois descent we have an equivalence $\mathrm{Coh}^G(X_L) \simeq \mathrm{Coh}(X)$ and therefore $\mathrm{D^b}(\mathrm{Coh}^G(X_L)) \simeq \mathrm{D^b}(X)$. A reasonable construction should give $\mathrm{D^b}(X_L)$ as the base change category of $\mathrm{D^b}(X)$. Generalising this example, one could start with any finite subgroup H of the group of autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X_L))$, define some triangulated category of H-linearised objects and then perform base change for this category. The question is then whether this encodes interesting geometric information.

We will use the example from geometry, namely the passage from $\mathrm{D^b}(X)$ to $\mathrm{D^b}(X_L)$, as our guide for a possible construction. We will often assume that the field extension L/K is finite, although some of the arguments do indeed generalise to arbitrary extensions.

The problem one faces in proposing a reasonable construction is that triangulated categories are not as rigid as, say, abelian categories. For the latter categories, as well as for additive ones without additional structure, there is in fact a well-known and fairly simple construction (see e.g. [1] or [38]), which gives the expected results if applied to e.g. the abelian category of sheaves on a scheme X. This construction is recalled in detail in Section 3.1. There is also a slightly different approach which appears e.g. in [67] and which is structurally similar, but uses Ind-objects. We can avoid this more technical construction, mostly because we usually work with finite extensions. The reason why this approach cannot work for a triangulated category basically boils down to the fact that the cone is not functorial. To circumvent this problem we shall apply the construction to enhanced triangulated categories, i.e. categories where the cone is in fact functorial. We will therefore recall the basic definitions and properties of (pretriangulated) differential graded categories and introduce base change for them.

After having established the necessary results we present the definition of base change: The basic idea would be to choose an enhancement $\mathcal{A}$ of $\mathcal{T}$, define base change for the enhancement and consider the homotopy category of the base change category. However, this simple direct approach does not work and one has to make the definition slightly more involved. We then prove our

Theorem 4 (Propositions 3.3.4, 3.3.5 and 3.3.7) *Let $\mathcal{T}$ be a triangulated category over K with a fixed enhancement. Then an L-linear triangulated category $\mathcal{T}_L$ can be constructed in a natural way. If X is a smooth projective variety over K and $\mathcal{T} = \mathrm{D^b}(X)$, then $\mathcal{T}_L$ is equivalent to $\mathrm{D^b}(X_L)$. If L/K is finite, then the last statement holds for any noetherian scheme X. Furthermore, if $\mathcal{C}$ is an abelian category with enough injectives and with generators, then $\mathrm{D^b}(\mathcal{C})_L$ is equivalent to $\mathrm{D^b}(\mathcal{C}_L)$.*

Our construction relies on the choice of an enhancement of $\mathcal{T}$. Unfortunately, we were not able to prove that working with a different enhancement produces the same result. The result above therefore has to be read as a statement involving one specific enhancement.

Disregarding the just mentioned problem one can still ask how certain properties of our triangulated category behave under scalar extension. We consider one example: In [62] the notion of the dimension of a triangulated category was introduced. In the last section of Chapter 3 we consider its behaviour under base change and prove

Theorem 5 (Propositions 3.4.4 and 3.4.5) *Let $\mathcal{C}$ be an abelian category with enough injectives and with generators and let L/K be a finite Galois extension. Then $\dim(\mathrm{D^b}(\mathcal{C})_L) = \dim(\mathrm{D^b}(\mathcal{C}))$. One also has $\dim(\mathrm{D^b}(X_L)) = \dim(\mathrm{D^b}(X))$ for any noetherian scheme X.*

About the ground field: Since our goal in the first chapter is to produce derived equivalent conjugate complex K3 surfaces, we work in characteristic zero. In fact, we work with fields lying between $\overline{\mathbb{Q}}$ and $\mathbb{C}$. In the second chapter the characteristic is allowed to be finite, but working with Galois extensions we have, for the most part, to assume that the characteristic of the ground field is prime to the order of the Galois group. In the last chapter no assumptions on the characteristic are made.

Notations: We write (Q)Coh(X) for the category of (quasi-)coherent sheaves on a scheme X. If $\mathcal{C}$ is any additive category we write $Kom(\mathcal{C})$ for the category of complexes over $\mathcal{C}$, $K(\mathcal{C})$ for the homotopy category and $\mathrm{D}(\mathcal{C})$ for the derived category. We use the usual notations for the various boundedness conditions, thus e.g. $\mathrm{D^b}(\mathrm{Coh}(X))$ is the bounded derived category. The latter will also be denoted by $\mathrm{D^b}(X)$. We will not use special symbols to denote derivation of functors. We will write $\mathrm{GL}^+(2, \mathbb{R})$ for the group of real 2×2-matrices with positive determinant.

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Chapter 1

Derived equivalent conjugate K3 surfaces

In this chapter we prove Theorem 1.3.6 which says that there exist derived equivalent non-isomorphic complex projective K3 surfaces, which are also conjugate to each other via an automorphism of the complex numbers. The proof is given in Section 1.3 through the construction of a certain curve in the moduli space of projective K3 surfaces of degree 12 over $\overline{\mathbb{Q}}$. In order to be able to do this we start with a presentation of some basic facts about K3 surfaces over an arbitrary field K of characteristic zero in Section 1.1. In particular, we present some classical results about ample line bundles on K3 surfaces. For a large part of this first section we restrict to $K = \mathbb{C}$ and recall the classical theorems about the surjectivity of the period map and the Global Torelli Theorem, which is one of the most important results in the theory: It gives a criterion for two K3 surfaces to be isomorphic in terms of the existence of a certain isomorphism of their second integral cohomology groups. We then deal with the Derived Torelli Theorem, which addresses the question when two complex K3 surfaces are derived equivalent, and recall what is known about the possible Fourier–Mukai partners of a given K3 surface. In Section 1.2 we consider moduli spaces of polarised K3 surfaces over $\mathbb{C}$ and over $\overline{\mathbb{Q}}$ and study certain maps, so called Mukai involutions, on these spaces. We prove that these maps are in fact morphisms and consider their fixed point locus in the case where the polarisation is of degree 12. In the following section we prove our main theorem and show that it can also be used to produce higher-dimensional derived equivalent conjugate varieties. In Section 1.4 we show that an analogue of Theorem 1.3.6 holds for abelian surfaces as well. Finally, in Section 1.5 we give an alternative proof of Proposition 1.2.8.

1.1 K3 surfaces

Let K be a field of characteristic zero, which in the following mostly will be the field of algebraic or the field of complex numbers.

Definition 1.1.1. A *K3 surface* is a smooth two-dimensional projective variety X over K such that $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Although non-projective surfaces play an important part in the theory over $K = \mathbb{C}$, we will ignore those and work under the projectivity assumption.

Example 1.1.2. The following are some basic examples of K3 surfaces.
(1) Consider the hypersurface given by a generic polynomial of degree four in $\mathbb{P}^3$. It is smooth by assumption. The long exact cohomology sequence associated to the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

gives that $H^1(X, \mathcal{O}_X) = 0$. Finally, the adjunction formula gives

$$\omega_X \simeq \omega_{\mathbb{P}^3} \otimes \mathcal{O}(X)_{|X} = (\mathcal{O}(-4) \otimes \mathcal{O}(4))_{|X} \simeq \mathcal{O}_X.$$

An explicit example of such a *quartic K3 surface* is e.g. given by the *Fermat quartic* $\left\{\underline{x} \in \mathbb{P}^3 \ : x_0^4 + \ldots + x_3^4 = 0\right\}$.
(2) In a similar fashion one proves that a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^4$ or the intersection of three quadrics in $\mathbb{P}^5$ is a K3 surface.
(3) An *elliptic K3 surface* is given by a morphism $X \longrightarrow \mathbb{P}^1$ such that almost all fibres are elliptic curves.
(4) Let A be an abelian surface (see Section 1.4). The involution map ι on A sending a point x to $-x$ has 16 fixed points. Blowing-up A in these 16 points gives a surface $\widetilde{A}$ and the involution ι induces an involution $\widetilde{\iota}$ on $\widetilde{A}$. It can then be shown that the quotient $X = \widetilde{A}/\widetilde{\iota}$ is a K3 surface. It is called the *Kummer surface* associated to A. In fact, one can also construct X by taking the minimal resolution of the 16 singular points of the quotient A/ι. The K3 surface X is sometimes denoted by $\mathrm{Kum}(A)$.

If X is a K3 surface, then an ample line bundle L on X will be called a *polarisation*. The self-intersection number (L, L) of such an L is called the *degree* of the polarised K3 surface (X, L), so e.g. the intersection of a quadric and a cubic in $\mathbb{P}^4$ has degree six. A polarisation is *primitive* if L is not a power of any line bundle on X. Using the first assertion of the next lemma one can see that, for example, a polarisation with self-intersection number 12 is automatically primitive.

Lemma 1.1.3. *The self-intersection number $(L, L) = 2d$ is even for any line bundle L on X. An ample line bundle L is effective and its Hilbert polynomial is given by $h_L(t) = dt^2 + 2$.*

Proof. Using Serre duality, the triviality of the canonical sheaf and the assumption $H^1(X, \mathcal{O}_X) = 0$ we immediately derive that $\chi(X, \mathcal{O}_X) = 2h^0(X, \mathcal{O}_X) = 2$. Applying the Riemann–Roch theorem it is then easy to see that

$$\chi(X, L) = \frac{1}{2}((L, L) - (L, \omega_X)) + \chi(X, \mathcal{O}_X) = \frac{1}{2}(L, L) + 2$$

and hence (L, L) is even for any line bundle L on X.

As to the second assertion: First, note that $H^2(L) = 0$, because $H^2(L) = H^0(L^{-1})$ and the anti-ample line bundle L^{-1} does not have global sections.

Rewriting the above formula and using that $(L,L)>0$, we conclude that L has global sections. Its Hilbert polynomial is given by

$$h_L(t) = (\frac{1}{2}(L,L))t^2 - (\frac{1}{2}(L,\omega_X))n + \chi(X,\mathcal{O}_X) = (\frac{1}{2}(L,L))t^2 + 2$$

and this concludes the proof. □

Remark 1.1.4. We will also need results from [63], which, in particular, give that if L is an ample line bundle on a K3 surface, then L^n is generated by global sections for $n \geq 2$ and is very ample for $n \geq 3$.

A *family of K3 surfaces* is a proper and flat morphism $\pi : \mathcal{X} \longrightarrow S$ over a scheme S, which as all schemes in this chapter will be assumed to be of finite type over the field in question, such that the geometric fibres of π are K3 surfaces. A family of (primitively) polarised K3 surfaces is given by a map π as above together with a line bundle $\mathcal{L}$ on $\mathcal{X}$ which defines a (primitive) polarisation restricted to each geometric fibre. Note that $\mathcal{L}^n$ is relatively very ample over S for $n \geq 3$.

For the rest of this section we will work over the complex numbers. Consider a complex K3 surface X and its singular cohomology groups. Since X is projective (in fact, for what follows it is sufficient for X to be Kähler and the non-projective complex K3 surfaces have this property by [66]), we can use Hodge theory and by the Hodge decomposition we see that $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) = 0$, since by assumption $H^{0,1}(X) = H^1(X,\mathcal{O}_X) = 0$. Therefore, using the cohomology sequence associated to the exponential sequence we conclude that $H^1(X,\mathbb{Z})$ is also 0. Clearly $H^0(X,\mathbb{Z}) = H^4(X,\mathbb{Z}) = \mathbb{Z}$ and it is also possible to show that $H_1(X,\mathbb{Z}) = 0$ and hence by Poincaré duality $H^3(X,\mathbb{Z}) = 0$. The only remaining cohomology group is thus $H^2(X,\mathbb{Z})$. The intersection pairing on $H^2(X,\mathbb{R})$ endows it with a structure of a lattice, i.e. a free abelian group of finite rank with a non-degenerate symmetric bilinear integer-valued form, abstractly isomorphic to the lattice

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Here, U is the *hyperbolic plane*, i.e. the free module $\mathbb{Z}^2$ with bilinear form given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $E_8(-1)$ is the standard root lattice changed by a sign, cf. [3, I.2.7]. Thus, Λ is isomorphic to the unique unimodular even lattice of signature $(3, 19)$. Here, even means that for any element α in the lattice the integer (α,α) is even, and unimodular means that the canonical embedding of Λ into its dual $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\mathbb{Z})$ is an isomorphism.

Definition 1.1.5. The lattice $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ is the *K3 lattice.*

To shorten notation we will write Λ_K for the vector space $\Lambda \otimes_{\mathbb{Z}} K$ for $K = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

There is a weight two Hodge structure on $H^2(X,\mathbb{Z})$ given by

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X).$$

Since

$$H^{2,0}(X) = H^0(X,\omega_X) \simeq H^0(X,\mathcal{O}_X)$$

is one-dimensional, we can write $H^{2,0}(X) = \mathbb{C}\sigma$ for some holomorphic two-form σ. The line $H^{2,0}(X)$ determines the Hodge structure, since $H^{0,2}(X)$ is complex conjugate to it and $H^{1,1}(X)$ is orthogonal to $H^{2,0}(X) \oplus H^{0,2}(X)$ with respect to the intersection pairing. We have

$$\mathrm{Pic}(X) = \mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X).$$

The first equality follows from the injectivity of $c_1 : H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X,\mathbb{Z})$ and the second from the fact that the second integral cohomology group of a K3 surface is torsion-free. The rank of $\mathrm{Pic}(X)$ is denoted by $\rho(X)$. The maximal Picard rank of a K3 surface in characteristic zero is 20 (in characteristic p it is 22) and the minimal one is, of course, 1 (recall that we work under the projectivity assumption). A generic K3 surface is in fact of Picard rank 1. One defines the *transcendental lattice* $T(X)$ to be the orthogonal complement of $\mathrm{Pic}(X)$ in $H^2(X,\mathbb{Z})$. It inherits a weight two Hodge structure.

To state one of the major results in the theory of K3 surfaces, the Global Torelli Theorem, recall that if X and X' are complex K3 surfaces, then a *Hodge isometry* between $H^2(X,\mathbb{Z})$ and $H^2(X',\mathbb{Z})$ is a lattice isomorphism f which respects the quadratic forms and the Hodge structures, where the latter in this case amounts to saying that $(f \otimes \mathbb{C})(H^{2,0}(X)) \subset H^{2,0}(X')$.

Theorem (Global Torelli): Let X and X' be two complex K3 surfaces. There exists an isomorphism $X \simeq X'$ over $\mathbb{C}$ if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(X',\mathbb{Z})$.

This is the weak version. The stronger one addresses the question when the Hodge isometry is induced by an isomorphism, which actually turns out to be unique. The history of the theorem is long with main contributions by Burns and Rapoport [18] and by Piateckii-Shapiro and Shafarevich [56], see the last remark in Ch. VIII in [3].

There is a version of the above theorem when one does not look for isomorphic K3 surfaces, but for derived equivalent ones. We first need to recall the

Definition 1.1.6. Two complex K3 surfaces X and Y are called *Fourier–Mukai partners* (abbreviated FM-partners), or *derived equivalent*, if there exists a $\mathbb{C}$-linear exact equivalence between their derived categories of coherent sheaves.

Of course, the definition of derived equivalence makes sense for arbitrary varieties. For information on the subject see the book [33].

We also have to introduce the *Mukai lattice* $\widetilde{H}(X,\mathbb{Z})$, which is the abelian group $H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$ with the bilinear form given by

$$\langle (a,l,b),(a',l',b') \rangle := ll' - ab' - ba'.$$

This pairing is called the *Mukai pairing.* Of course, this is nothing but the usual intersection product pairing, up to signs. Using the above description of $H^2(X,\mathbb{Z})$ we see that

$$\widetilde{H}(X,\mathbb{Z}) \simeq \Lambda \oplus U$$

and hence the signature of this lattice is $(4,20)$. There is a weight two Hodge structure on $\widetilde{H}(X,\mathbb{Z})$ given by

$$\widetilde{H}^{2,0} := H^{2,0}(X),\ \widetilde{H}^{0,2} := H^{0,2}(X) \text{ and } \widetilde{H}^{1,1} := H^{1,1}(X) \oplus H^0(X,\mathbb{C}) \oplus H^4(X,\mathbb{C}).$$

With this definition the embedding $H^2(X,\mathbb{Z}) \hookrightarrow \widetilde{H}(X,\mathbb{Z})$ clearly preserves the Hodge structures.

We can now state the

Theorem (Derived Torelli): The following conditions are equivalent:
(1.) Two complex K3 surfaces X and Y are derived equivalent.
(2.) There exists a Hodge isometry between $T(X)$ and $T(Y)$.
(3.) There exists a Hodge isometry between $\widetilde{H}(X,\mathbb{Z})$ and $\widetilde{H}(Y,\mathbb{Z})$.
(4.) Y is a fine moduli space of stable sheaves on X with respect to a certain polarisation.

Proof. Mukai proved in [45] that (1.) implies (3.). In [53] the converse and the equivalence of (3.) and (4.) were shown. The equivalence of (2.) and (3.) is proved as follows: It is clear that any Hodge isometry between the full cohomology groups induces a Hodge isometry between the transcendental lattices, because $T(X)$ is the smallest primitive sublattice of $H^2(X,\mathbb{Z})$ containing $H^{2,0}(X)$ after complexification. For the converse one uses [50, Thm. 1.14.1] which states that an isometry of the primitive even sublattice $T(X)$ can be extended to an isometry of $\widetilde{H}(X,\mathbb{Z})$, since the orthogonal complement $T(X)^\perp$ of $T(X)$ in $\widetilde{H}(X,\mathbb{Z})$ contains the hyperbolic plane $H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$. □

Remark 1.1.7. A priori, 'stable' in (4.) means 'Gieseker stable', but, in fact, one can replace 'Gieseker stable' by 'μ-stable' and 'sheaves' by 'locally free sheaves', cf. [34, Prop. 4.1]. For the definition and properties of the different notions of stability of sheaves see the book [35].

Remark 1.1.8. In some cases it is possible to compute the number of non-isomorphic Fourier–Mukai partners of a given K3 surface X (cf. [30, Cor. 2.7]):
(a) If $\rho(X) \geq 12$, then X does not have any non-trivial FM-partners.
(b) If $\rho(X) \geq 3$ and the determinant of $\mathrm{Pic}(X)$ is square free, then X does not have any non-trivial FM-partners.
(c) If X is an elliptic K3 surface (cf. (3) in Example 1.1.2) with a section, then X does not have any FM-partners.
(d) If $\rho(X) = 1$, so $\mathrm{Pic}(X) = \mathrm{NS}(X) = \mathbb{Z}H$ with $H^2 = 2d > 0$, then the number of non-isomorphic FM-partners of X is equal to $2^{\tau(d)-1}$, where $\tau(d)$ is the number of distinct primes dividing d (cf. also [51, Prop. 1.10]).

The proof is via lattice theory. For example, (c) follows from [50, Thm. 1.14.1]: The Picard group of an elliptic K3 surface with a section contains a hyperbolic plane, namely the span of a fibre and the section. Therefore any Hodge isometry of the transcendental lattice extends to a Hodge isometry of the second cohomology group and the Global Torelli Theorem applies.

The K3 surfaces arising in (d) can in fact be described fairly explicitly: One uses part (4.) of the Derived Torelli Theorem and it turns out that it is possible to write down the numerical invariants of the moduli spaces in question. Apart from the above results there is the following: It was proved in [51] and later in [68] that for a given natural number N there are at least N pairwise non-isomorphic derived equivalent K3 surfaces. These surfaces are elliptic, their Picard rank is 2 and, in contrast to the Picard rank 1 case, they do not necessarily have polarisations of the same degree.

The above shows that there are interesting geometric connections between derived equivalent K3 surfaces. There is a different operation for complex K3 surfaces, which is somewhat arithmetic in nature:

Definition 1.1.9. Let X be a complex projective K3 surface and let σ be an automorphism of the complex numbers. We define the *conjugate K3 surface* X^σ by the fibre product

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}). \end{tikzcd}$$

Remark 1.1.10. Clearly this definition can be applied to any complex variety. Note that if the variety in question is the vanishing set of some polynomials f_i, $i = 1, \dots, k$, then the conjugate variety is precisely the vanishing set of $\sigma(f_i)$. Here, for a polynomial $f = \sum a_J \underline{x}_J$, $a_J \in \mathbb{C}$, we denote by $\sigma(f)$ the polynomial $\sum \sigma(a_J)\underline{x}_J$. For example, if the polynomials defining the variety all have rational coefficients, then conjugation by any σ is the identity.

In general the process of conjugation can change the homotopy type as was shown in [64], where Serre constructs two conjugate complex projective varieties with different fundamental groups. In our special case the change in geometry caused by conjugation is more subtle. Note that e.g. the Picard groups of X and X^σ are the same.

Question: Is it possible to determine the number of non-isomorphic conjugate surfaces of a given K3 surface X? Clearly, the field of definition of X will be a part of the answer.

Remark 1.1.11. The K3 surfaces X and X^σ will in general be non-isomorphic as schemes over $\mathbb{C}$ but clearly they are isomorphic over $K = \mathbb{Q}$ and thus there is a $\mathbb{Q}$-linear equivalence $\mathrm{D^b}(X) \simeq_{\mathbb{Q}} \mathrm{D^b}(X^\sigma)$.

Theorem 1.3.6 below gives an affirmative answer to the following

Question: Do there exist non-isomorphic derived equivalent complex K3 surfaces which are also conjugate?

We will see below (Corollary 1.2.3) that it is possible to view derived equivalent K3 surfaces as elements in the orbit of the action of a certain discrete group on the moduli space of projective K3 surfaces (see e.g. [32]). By definition the group $\mathrm{Aut}(\mathbb{C})$ also acts on this moduli space. Thus, we would like to understand the connection between the actions of $\mathrm{Aut}(\mathbb{C})$ and the mentioned discrete group on the moduli space of projective K3 surfaces.

1.2 Moduli spaces of K3 surfaces and Mukai involutions

Let K be an algebraically closed field of characteristic 0. Using the general theory in [73] it is possible to construct a coarse quasi-projective moduli scheme $\mathcal{M}_{2d}^K$ for primitively polarised K3 surfaces of degree $2d$ over K as follows: Consider the Hilbert scheme Hilb_N^P representing subvarieties of $\mathbb{P}_K^N$ with Hilbert polynomial $P(x) := n^2dx^2 + 2$, where $n \geq 3$, d is a natural number (which should be thought of as $(\frac{1}{2}(L,L)))$ and $N = P(1) - 1$ (for some details on the Hilbert functor see the end of Section 1.3). Then there exists on open subscheme U of Hilb_N^P representing primitively polarised K3 surfaces together with an embedding into $\mathbb{P}_K^N$ and the GIT quotient $U/PGL(N+1)$ is the coarse moduli scheme in question.

For $K = \mathbb{C}$ there is a different construction, which shows that the moduli scheme of polarised complex K3 surfaces $\mathcal{M}_{2d}^{\mathbb{C}}$ is actually a 19-dimensional reduced, irreducible and normal space. To do this, we need to recall quite a few definitions.

Definition 1.2.1. A *marked K3 surface* is a K3 surface together with the choice of an isometry $\varphi : H^2(X,\mathbb{Z}) \simeq \Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.

Recall that $H^{2,0}(X) = \mathbb{C}\sigma$ for some holomorphic two-form σ. The *period* of a K3 surface is by definition the point $[\sigma] \in \mathbb{P}(H^2(X,\mathbb{C})) \simeq \mathbb{P}(\Lambda_{\mathbb{C}})$. To be more precise, $[\sigma]$ is an element of the *period domain*

$$\Omega = \left\{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid x^2 = 0, x\overline{x} > 0\right\}.$$

This space is an analytically open subset of the 21-dimensional quadric defined by the condition $x^2 = 0$. The *period map* assigns to a marked K3 surface (X,φ) its period point. The Global Torelli Theorem implies the

Corollary 1.2.2. *Two complex K3 surfaces X and X' are isomorphic if and only if there exist markings φ for X and φ' for X' such that the corresponding period points are equal. Equivalently, their period points lie in one orbit of the action of $O(\Lambda)$, where $O(\Lambda)$ is the group of isometries of Λ.* □

The Derived Torelli Theorem translates to

Corollary 1.2.3. *Two complex K3 surfaces X and Y are derived equivalent if their period points lie in one orbit of the action of $O(\widetilde{\Lambda})$.* □

One also has the following important

Theorem (Surjectivity of the period map) For any point x in Ω there exists a K3 surface X and a marking φ such that the period point of (X,φ) is x.

Similarly to the Global Torelli Theorem the proof of this result has a long history, cf. the last remark in Ch. VIII in [3].

We will be interested in the polarised version of the above. A (primitive) polarisation of degree $2d$ corresponds to a (primitive) vector h in Λ such that $h^2 = 2d$ and a marked polarised K3 surface is a marked surface such that the marking respects the polarisation. The orthogonal complement of h in Λ is isometric to the lattice

$$\Lambda_{2d} = \langle k \rangle \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2},$$

where $k^2 = -2d$. Since a polarisation is an element in $H^{1,1}(X)$, it is orthogonal to σ and hence the period point of a polarised K3 surface (X,L) lies in the *polarised period domain*

$$\Omega_{2d} = \Omega \cap \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}).$$

Now set

$$\Gamma(h) = \{g \in O(\Lambda) \mid g(h) = h\} \quad \text{and} \quad \Gamma_{2d} = \mathrm{im}(\Gamma(h) \longrightarrow O(\Lambda_{2d})),$$

where $O()$ denotes the group of isometries of the lattice in parentheses. Then, by the surjectivity of the period map and the Global Torelli Theorem we have the equality

$$\mathcal{M}_{2d}^{\mathbb{C}} = \Omega_{2d}/\Gamma_{2d}.$$

Clearly, this space is 19-dimensional, being the quotient of a 19-dimensional space by a discrete group. It is furthermore an irreducible reduced normal scheme, see [4].

We would like to see that $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ inherits all the above mentioned properties. A first step is the following

Lemma 1.2.4. $\mathcal{M}_{2d}^{\mathbb{C}} \simeq \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$.

Proof. Both schemes can be constructed as GIT-quotients and this is compatible with field extensions, cf. [47, Prop. 1.14]. □

Using the lemma we can now deduce the

Corollary 1.2.5. $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ *is an integral 19-dimensional normal scheme.*

Proof. Since $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$ is irreducible, so is $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ being its image under the projection. Further, $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is reduced, since $\mathcal{M}_{2d}^{\mathbb{C}}$ is and we can check this property using an affine cover. The statement about the dimension is clear.

To prove the normality we will use the following result from commutative algebra:
Let R and S be two K-algebras such that $R \otimes_K S$ is Noetherian. Then $R \otimes_K S$ is normal if and only if R and S are normal (this is a special case of [71, Thm. 6]).
Since normality is a local property, we may take an open affine subset $\mathrm{Spec}(A)$ in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. Then, by normality of $\mathcal{M}_{2d}^{\mathbb{C}}$, we have that $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is normal (and of course Noetherian) and hence A is. □

Remark 1.2.6. The normality can also be proved as follows: The scheme $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is the coarse moduli space of a smooth Deligne-Mumford stack (cf. [60]). It is thus normal being the quotient of a smooth scheme by a finite group (characteristic zero is needed here).

We will now study so called Mukai involutions on $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. We will use several results obtained in [69], where the action of the group of Mukai involutions on $\mathcal{M}_{2d}^{\mathbb{C}}$ was investigated.

Let (X, L) be a complex $2d$-polarised K3 surface with $L^2 := (L, L) = 2d = 2rs$ such that $\gcd(r, s) = 1$ and $r \leq s$. The *Mukai vector* $v(E)$ of a coherent sheaf E on X is by definition

$$v(E) = \mathrm{ch}(E) \cdot \sqrt{\mathrm{td}(X)}.$$

Thus, the numerical invariants of a sheaf E with Mukai vector

$$(a, l, b) = v \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

are given by $\mathrm{rk}(E) = a$, $c_1(E) = l$ and $\chi(E) = a + b$. We can consider the moduli space $M_X = M_X(v)$, where $v = (r, L, s)$ with r, L, s are as above. With these choices one has that M_X is a fine moduli space and, furthermore, a $2d$-polarised K3 surface, see [69]. We will use the same notation over $\overline{\mathbb{Q}}$. In this case M_X is also in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$, which follows from the fact that this is true over $\mathbb{C}$ and that any line bundle on the K3 surface $(M_X)_{\mathbb{C}}$ is already defined over $\overline{\mathbb{Q}}$, since $\mathrm{Pic}(X) = \mathrm{Pic}(X_{\mathbb{C}})$ for a K3 surface X over $\overline{\mathbb{Q}}$. For a proof of the last statement cf. [31, Prop. 5.4]. We define the *Mukai involution* g to be the map over $\overline{\mathbb{Q}}$ sending X to M_X. For the discussion of the fact that $g_{\mathbb{C}}$, and therefore also g, is indeed an involution cf. subsection 2.1 in [69].

Remark 1.2.7. If a complex K3 surface X has Picard rank one all Fourier–Mukai partners can be determined explicitly by describing the Mukai vectors v as above. To be more precise, any FM-partner of a complex $2d$-polarised K3 surface (X, L) is of the form $M_X(r, L, s)$ with r, L, s as above. Furthermore, $M_X(v) \ncong M_X(v')$ for $v \neq v'$. Compare statement (d) in Remark 1.1.8.

Proposition 1.2.8. *The Mukai involution g is a morphism.*

Proof. Consider the universal family $f : \mathcal{X} \longrightarrow U$, where U is the open subscheme of the Hilbert scheme used above. Note that U is reduced. The morphism f is projective and hence there exists a relative moduli space $\mathcal{M}(v) \longrightarrow U$ such that over $t \in U$ we have the moduli space $M_{\mathcal{X}_t}(v)$ (cf. [35, Thm. 4.3.7]). By construction there exists a polarisation $\widetilde{\mathcal{L}}$ on $\mathcal{M}(v)$. Its intersection number on the fibres is a quadratic multiple, say a, of the given degree $2d$. This can be seen by looking at K3 surfaces of Picard rank 1 and using that the intersection number is (locally) constant. Now, the étale sheafification of the relative Picard functor is representable by a scheme $\mathrm{Pic}_{\mathcal{M}(v)/U}$ (see e.g. [13, Ch. 8]) and the image of the morphism $f : U \longrightarrow \mathrm{Pic}_{\mathcal{M}(v)/U}$ defined by $\widetilde{\mathcal{L}}$ lies in the image of the closed immersion $[a] : \mathrm{Pic}_{\mathcal{M}(v)/U} \longrightarrow \mathrm{Pic}_{\mathcal{M}(v)/U}$ (where $[a]$ is the multiplication by a; it is a closed immersion by [60, Lem. 3.1.6]). We therefore have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\widetilde{f}"'] & \mathrm{Pic}_{\mathcal{M}(v)/U} \\ & \mathrm{Pic}_{\mathcal{M}(v)/U}. \arrow[u, "{[a]}"'] \end{tikzcd}$$

The morphism $\widetilde{f}$ defines an element of the étale Picard functor, which by definition is represented by a line bundle $\mathcal{L}'$ on the fibre product $\mathcal{M}(v)' := \mathcal{M}(v) \times_U U'$, for some étale covering $\pi : U' \longrightarrow U$, with the property that $\widetilde{\pi}^*(\widetilde{\mathcal{L}}) \simeq \mathcal{L}'^a$ ($\widetilde{\pi}$ is the natural projection). Thus, $\mathcal{M}(v)' \longrightarrow U'$ is a family of K3 surfaces with a polarisation of degree $2d$ and we therefore get a map $\alpha : U' \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. Using descent theory described in [24, Exp. VIII] we know that there exists a morphism $\beta : U \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ such that $\beta\pi = \alpha$ if and only if

α commutes with the two projections from $U' \times_U U'$, i.e. $\alpha p_1 = \alpha p_2$. But the latter condition is clear for closed, and hence for all (everything is reduced), points by the fact that α is the classifying map of the family and the K3 surfaces over the closed points of a fibre of π are all isomorphic. Thus we have a map $\beta : U \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ sending t to $M_{\mathcal{X}_t}(v)$. Since α is equivariant, it descends to a morphism h from the GIT-quotient $U/PGL = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ to $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ so that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\beta"] \arrow[d] & \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \\ \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}. \arrow[ur, dotted, "h"] & \end{tikzcd}$$

By definition we have that $g = h$ and thus g is a morphism. $\square$

Remark 1.2.9. Clearly the same proof works for $\mathcal{M}_{2d}^{\mathbb{C}}$ (and any other algebraically closed field of characteristic zero). In particular, we shall need $g_{\mathbb{C}} : \mathcal{M}_{2d}^{\mathbb{C}} \longrightarrow \mathcal{M}_{2d}^{\mathbb{C}}$.

It follows from the Derived Torelli Theorem that $X_{\mathbb{C}} = X \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$ and $M_{X_{\mathbb{C}}}(v) = M_X \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$ are FM-partners.

From now on we will consider the case $2d = 12$. In this case there is only one Mukai involution sending a K3 surface X to the moduli space $M_X = M_X(2, l, 3)$. This involution will be denoted by g. The first step is to investigate its fixed point locus. It was proved in [69] that over $\mathbb{C}$ the fixed point locus of $g_{\mathbb{C}}$ contains a divisor D. We will now prove the

Proposition 1.2.10. *There exists a divisor in the fixed point locus Fix(g) of the morphism* $g : \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$.

Proof. The fixed point locus of a morphism can be defined as the intersection of the diagonal and the graph which are both defined over $\overline{\mathbb{Q}}$. Since this construction commutes with base change we have that $\mathrm{Fix}(g_{\mathbb{C}}) = \mathrm{Fix}(g) \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$. Therefore $\mathrm{Fix}(g)$ has to contain a divisor. $\square$

Remark 1.2.11. The divisor in $\mathrm{Fix}(g_{\mathbb{C}})$ corresponds to K3 surfaces whose Picard lattice contains a certain rank two nondegenerate even lattice. From [43, Cor. 1.9 and Cor. 2.5] we know that there exists a K3 surface X of Picard rank 20 in D. By [65, Thm. 6] X can be defined over a number field and in particular over $\overline{\mathbb{Q}}$. In fact, the points of D defined over $\overline{\mathbb{Q}}$ are dense in D which gives another proof of the above proposition.

1.3 A special curve in $\mathcal{M}_{12}^{\overline{\mathbb{Q}}}$

In this section we will prove Theorem 1.3.6. The basic idea of the proof is to find a curve C in the moduli space of K3 surfaces over $\overline{\mathbb{Q}}$ of degree 12 which is invariant under the unique Mukai involution described above. The induced automorphism of the function field $K(C)$ will allow us to produce two non-isomorphic derived equivalent K3 surfaces over $\mathbb{C}$. Since this automorphism of $K(C)$ extends to an automorphism of $\mathbb{C}$ these two K3 surfaces will also be conjugate.

We want to construct an irreducible g-invariant curve in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. The strategy is rather simple: Take a curve in the quotient space $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}/\langle \mathrm{id}, g\rangle$ and pull it back to $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. The curve C we get will clearly be g-invariant. The irreducibility will be achieved by using Bertini's theorem.

We first recall the following

Proposition 1.3.1. *The quotient* $\mathcal{K} = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}/\langle \mathrm{id}, g\rangle$ *is an algebraic variety. The projection map* $\pi : \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \longrightarrow \mathcal{K}$ *is finite and surjective.*

Proof. Since $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is quasi-projective, this follows from [46], pp. 66–69. □

To ensure that C is irreducible, we want it to be connected and regular. The first property is provided by the following

Proposition 1.3.2. *Let A be an irreducible curve in $\mathcal{K}$ which is not contained in the image of the fixed point locus of g but intersects it in at least one point. Then the pullback curve $C = \pi^{-1}(A) = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathcal{K}} A$ is connected. Furthermore, g acts non-trivially on C.*

Proof. Assume the converse, then there exist disjoint closed non-empty subsets $C = W_1 \bigsqcup W_2$. Since A intersects the fixed point locus, there exists a point $x \in A$ whose reduced fibre is precisely one point y. We may assume that $y \in W_1$. The map π is finite, therefore proper, and hence $\pi(W_1), \pi(W_2)$ are closed in A. Since A is irreducible and both sets are non-empty, we must have $\pi(W_1) = \pi(W_2)$. However, this is impossible, because $\pi(W_2)$ does not contain x. The last assertion is obvious. □

Next we have to make sure that C is regular. To do this we use Bertini's theorem: Let A be given as an intersection of hyperplanes. If these hyperplanes are generic, then A is regular away from the singularities of $\mathcal{K}$. In order to control these we need the following

Lemma 1.3.3. *Let R be a normal integral domain, let H be a finite group of automorphisms of R and let R^H be the ring of invariants. Then R^H is normal.*

Proof. If $z \in Q(R^H)$ is integral over R^H, it is also integral over R and hence $z \in R$, since R is normal. On the other hand, $h(z) = z$ for all $h \in H$ and therefore $z \in R^H$. Thus, R^H is normal. □

Since we know $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ to be normal, it follows from the lemma that $\mathcal{K}$ is normal as well. Thus, if A is generic among those curves intersecting the fixed point locus in a generic point of the divisor, it will be regular and the same will hold for C (cf. [27, Ch. III, Cor. 10.9]). Thus, we have proved

Proposition 1.3.4. *There exists a g-invariant connected and regular, hence irreducible, curve C (on which g acts non-trivially) in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$.* □

However, $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is just a coarse moduli space and thus a priori we do not have a family over C. This problem is avoided by the following

Proposition 1.3.5. *There exists a family* $\mathcal{X}' \longrightarrow C'$ *over an irreducible curve* C' *such that the classifying map of this family is a finite surjective morphism* $C' \longrightarrow C$. *Denoting the function fields* $K(C)$ *resp.* $K(C')$ *by* $\mathbb{K}$ *resp.* $\mathbb{L}$ *we furthermore have that the inclusion* $\mathbb{K} \longrightarrow \mathbb{L}$ *is a Galois extension.*

Proof. The first statement is well-known, cf. e.g. [73, Thm. 9.25] (the general idea is that the wanted curve lives in the Hilbert scheme). Denote the covering curve by $\widetilde{C}$. Considering function fields we have a finite (and of course separable) extension $\mathbb{K} = K(C) \longrightarrow \mathbb{L}' := K(\widetilde{C})$. If this extension is not normal, we can take the normal closure $\mathbb{L}$ of $\mathbb{L}'$ to get a Galois extension $\mathbb{K} \longrightarrow \mathbb{L}$. Geometrically this just corresponds to a finite surjective morphism from a new curve $C' \longrightarrow \widetilde{C}$ and a (pullback-)family over C'. Hence the result. $\square$

The morphism g induces an automorphism of $\mathbb{K}$ which can be lifted to an automorphism $\widetilde{g}$ of $\overline{\mathbb{K}} = \overline{\mathbb{L}}$ (see e.g. [74, Thm. 6]). Considering the composition of $\widetilde{g}$ with the inclusion $\mathbb{L} \longrightarrow \overline{\mathbb{K}}$ and using the normality of $\mathbb{L}$, we see that we get an automorphism g' of $\mathbb{L}$ which clearly extends g. This automorphism then gives an automorphism of the curve C'. Since g' is an extension of g, it has the same geometric interpretation, namely sending a fibre X to a K3 surface which is isomorphic to $M_X(v)$.

The geometric fibre of the generic point of C' is a K3 surface and therefore the generic fibre itself is as well (use [27, Ch. III, Prop. 9.3]). Denote this K3 surface over $\mathbb{L}$ as $X_{\mathbb{L}}$. Base change via g' gives a second K3 surface over $\mathbb{L}$ which by construction is $\mathrm{M}_{X_{\mathbb{L}}}(2, l, 3) =: X'_{\mathbb{L}}$. Now fix an embedding i of $\mathbb{L}$ into $\mathbb{C}$. Denoting the induced action of g' on $\mathrm{Spec}(\mathbb{L})$ by g'^*, we have the following diagram:

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[r] \arrow[d] & X_{\mathbb{L}} \arrow[d] & X'_{\mathbb{L}} \arrow[l] \arrow[d] & X'_{\mathbb{C}} \arrow[l] \arrow[d] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r, "i^*"] & \mathrm{Spec}(\mathbb{L}) & \mathrm{Spec}(\mathbb{L}) \arrow[l, "g'^*"'] & \mathrm{Spec}(\mathbb{C}). \arrow[l, "i^*"']
\end{tikzcd}$$

Extending the automorphism g' of $\mathbb{L}$ to $\widehat{g} \in \mathrm{Aut}(\mathbb{C})$ we see that $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are conjugate via $\widehat{g}$. Clearly $X'_{\mathbb{C}} \simeq M_{X_{\mathbb{C}}}(2, l, 3)$ and thus $X'_{\mathbb{C}}$ is not isomorphic to $X_{\mathbb{C}}$ for a generic $X_{\mathbb{C}}$. Note that by construction X and X' are not isomorphic over $\mathbb{L}$, but they might become isomorphic over $\mathbb{C}$ and this is why the 'generic' assumption is needed. Hence, $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are Fourier–Mukai partners. We now derive the

Theorem 1.3.6. *There exist a complex projective K3 surface* X *and an automorphism* $\sigma \in \mathrm{Aut}(\mathbb{C})$ *with the property that the conjugate K3 surface* X^σ *is a non-isomorphic complex K3 surface and there exists a* $\mathbb{C}$*-linear equivalence* $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X^\sigma)$.

Proof. Set $X := X_{\mathbb{C}}$ and $\sigma := \widehat{g}$. $\square$

Remark 1.3.7. One might ask whether the phenomenon described above is at all special, since a priori it might be that any Fourier–Mukai partner of a complex K3 surface X is obtained by conjugation and/or vice versa. However, this is not the case as illustrated by the following: A generic quartic X in $\mathbb{P}^3_{\mathbb{C}}$ given by a polynomial P has Picard rank 1 and no non-trivial Fourier–Mukai partners by part (d) in Remark 1.1.8. However, conjugating X with a generic automorphism σ of $\mathbb{C}$ will produce a non-isomorphic conjugate quartic X^σ, since conjugation in this case amounts to applying σ to the coefficients of P. Hence a non-isomorphic conjugate of a K3 surface need not be a Fourier–Mukai partner.

It should also be true that there exists a complex K3 surface of Picard rank 1 having a Fourier–Mukai partner which is not a conjugate surface. By (d) in Remark 1.1.8 it would e.g. be enough to have a K3 surface defined over $\mathbb{Q}$ (so it remains fixed under conjugation) which has Picard rank 1 and is of degree $2d$ with d having several prime divisors. In [72] the author gives an explicit example of a quartic K3 surface with Picard rank 1 which is defined over $\mathbb{Q}$. Hopefully the techniques in [72] can be used more generally to produce a K3 surface of sufficiently high degree.

Note that there are examples of derived equivalent K3 surfaces which cannot be conjugate: These are the elliptic K3 surfaces of Picard rank 2 mentioned after Remark 1.1.8 and they have non-isometric Picard lattices. It would be very interesting to have a K3 surface of e.g. Picard rank 2 in the above theorem, because this would give an example of non-generic derived equivalent K3 surfaces with isometric Picard lattices.

Remark 1.3.8. We chose to work with K3 surfaces of degree 12 because it simplifies the exposition. The statements we needed from [69], in particular that the fixed point locus contains a divisor, hold more generally for degree $4p$, where p is an odd prime such that p is not a square modulo 4 and 2 is not a square modulo p. Even more generally, there are results in [69] describing conditions on the involution which ensure that its fixed point locus contains a divisor, thus giving us the possibility to use degrees with several prime factors.

We can use the above theorem to produce higher-dimensional examples of derived equivalent conjugate varieties. To do this recall that given a projective scheme W over $\mathbb{C}$ with an embedding $W \subset \mathbb{P}^r_{\mathbb{C}}$, a numerical polynomial $P(t) \in \mathbb{Q}[t]$ and a scheme S over $\mathbb{C}$ one defines

$$\underline{\mathrm{Hilb}}^W_{P(t)}(S) = \{Z \subset Y \times S \mid Z \text{ proper and flat}/S,\ P(Z_s) = P\ \forall\, s \in S\}\,.$$

This gives a contravariant functor from the category of schemes over $\mathbb{C}$ to the category of sets. This functor is representable by a projective scheme, cf. [35, Ch. 2]. If $P(t) = n$ is constant, then we will denote the functor by $\underline{\mathrm{Hilb}}^n_W$ and the scheme representing it, the *Hilbert scheme*, by $\mathrm{Hilb}^n(W)$ (of course, this construction works over other fields than $\mathbb{C}$).

For a K3 surface X the Hilbert schemes $X^{[n]} := \mathrm{Hilb}^n(X)$ are so called *irreducible holomorphic symplectic varieties* or *hyperkähler varieties*. This, for us, means that $X^{[n]}$ is a simply-connected projective variety such that the space of global sections of $\Omega^2_{X^{[n]}}$ is generated by a closed non-degenerate holomorphic two-form. The dimension of $X^{[n]}$ is $2n$. We will need the following easy

Lemma 1.3.9. *Let X be a complex projective variety. Then we have the equality $(X^{[n]})^\sigma = (X^\sigma)^{[n]}$ for any automorphism σ of $\mathbb{C}$.*

Proof. First, note that the map $f \longmapsto f^\sigma$ defines an isomorphism $\mathrm{Hom}(S,T) \simeq \mathrm{Hom}(S^\sigma, T^\sigma)$ for schemes S, T over $\mathbb{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, and, writing τ for σ^{-1}, the equality $(S^\sigma)^\tau \simeq S$ holds. Furthermore we also have $(S \times T)^\sigma \simeq S^\sigma \times T^\sigma$.

The claimed equality now follows from

$$\mathrm{Hom}(S, \mathrm{Hilb}^n(X^\sigma)) = \{Z \subset S \times X^\sigma \mid Z \,\text{proper}\,\&\,\text{flat}/S,\ P(Z_s) = n \,\forall\, s \in S\} \simeq$$
$$\simeq \{Z^\tau \subset S^\tau \times X \mid Z^\tau \text{ proper}\,\&\,\text{flat}/S^\tau,\ P(Z^\tau_s) = n \,\forall\, s \in S\} =$$
$$= \mathrm{Hom}(S^\tau, \mathrm{Hilb}^n(X)) \simeq \mathrm{Hom}(S, \mathrm{Hilb}^n(X)^\sigma),$$

since an object representing a functor is unique up to isomorphism. □

Now recall from [57, Prop. 8] that if we have an equivalence $\mathrm{D^b}(X) \simeq \mathrm{D^b}(Y)$ of two smooth projective surfaces X and Y, then there is also an equivalence $\mathrm{D^b}(\mathrm{Hilb}^n(X)) \simeq \mathrm{D^b}(\mathrm{Hilb}^n(Y))$. Thus, we derive the

Theorem 1.3.10. *There exist a hyperkähler variety Y and an automorphism σ of $\mathbb{C}$ such that Y and Y^σ are derived equivalent.*

Proof. Set $Y = \mathrm{Hilb}^n(X)$ with X as in Theorem 1.3.6 and use the lemma. □

Remark 1.3.11. If X and Y are non-isomorphic K3 surfaces, then it is not true in general that $\mathrm{Hilb}^n(X)$ and $\mathrm{Hilb}^n(Y)$ are also non-isomorphic for all n, cf. [75, Ex. 7.2] where the author gives an example such that the Hilbert schemes are indeed isomorphic for $n = 2$ and $n = 3$. Thus, in the above theorem we might have produced isomorphic Fourier–Mukai partners. However, we believe that the examples in [75] are rather special and that the theorem will indeed produce non-isomorphic Hilbert schemes in the general case.

1.4 Abelian surfaces

Recall that an *abelian surface* A over a field K of characteristic zero is by definition a complete algebraic surface over K with a group law $m : A \times A \longrightarrow A$ such that m and the inverse map are both morphisms of varieties. Such a surface is necessarily smooth, projective and it is commutative as a group (cf. e.g. [46]).

The *dual abelian surface* is the variety $\mathrm{Pic}^0(A)$, denoted by $\widehat{A}$. Given an ample line bundle L on A, there is an isogeny, i.e. a surjective morphism with finite kernel, $\phi_L : A \longrightarrow \widehat{A}$ defined by $x \longmapsto t_x^* L \otimes L^{-1}$, where t_x is the translation by x map. The kernel of this isogeny is isomorphic to $(\mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z})^2$ for some positive integers d_i with the property $d_1 | d_2$. The vector (d_1, d_2) is called the *type* of the line bundle L. A *polarised abelian surface* is a pair (A, L), where L is ample on A and the type of such a surface is by definition the type of L. A polarisation L is *principal* if and only if it is of type (1,1), i.e. the isogeny defined by L is an isomorphism.

Recall that for any algebraically closed field K of characteristic 0 there exists a coarse moduli space $\mathcal{A}_{(1,t)}$ for polarised abelian surfaces of type $(1, t)$. This moduli space is a quasi-projective normal threefold (see e.g. [47]).

Using [7, Thm. 1.1] we know that there exists an involution f_K on $\mathcal{A}_{(1,t)}$ sending a polarised abelian surface (A, L) to the dual polarised abelian surface $(\widehat{A}, \widehat{L})$. Here, the dual polarisation is e.g. defined by demanding that its pullback under the isogeny ϕ_L induced by L is tL. For other descriptions of $\widehat{L}$ (in the case where the ground field is $\mathbb{C}$) see [6].

In the case $K = \mathbb{C}$ we use [23, Thm. 3.4] to deduce that the fixed point locus of $f_{\mathbb{C}}$ contains at least one divisor. With the same arguments as in Proposition

1.2.10 we see that $\mathrm{Fix}(f_{\overline{\mathbb{Q}}})$, the fixed point locus over $K = \overline{\mathbb{Q}}$, contains a divisor as well. The techniques in Section 1.3 then apply and give an abelian surface A and a $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $\widehat{A} \simeq A^\sigma$. By the classical results of Mukai, see [44], A and $\widehat{A}$ are always derived equivalent and therefore A is an abelian surface such that its non-isomorphic Fourier–Mukai partner $\widehat{A}$ is a conjugate surface. Thus we have proved the

Theorem 1.4.1. *There exist an abelian surface A and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that the non-isomorphic Fourier–Mukai partners A and $\widehat{A}$ are conjugate abelian surfaces.* □

Remark 1.4.2. Similarly to the case of K3 surfaces one can associate a hyperkähler variety $K_n(A)$, $n \geq 1$, to any abelian surface A. It is of dimension $2n$ and is called the *generalised Kummer variety* of A. The name stems from the equality $K_1(A) = \mathrm{Kum}(A)$. One could hope to use the above theorem to produce yet another example of derived equivalent conjugate hyperkähler varieties. However, in the K3-case we used the fact that derived equivalent K3 surfaces have derived equivalent Hilbert schemes. Unfortunately, a similar result is missing for the generalised Kummer varieties. In [48] the author gives an example which shows that the generalised Kummer varieties of derived equivalent abelian surfaces need not be birational. It was shown in [29] (cf. also part (a) in Remark 1.1.8, since the Picard rank of a Kummer surface is at least 16) that for any two abelian surfaces A and B one has

$$\mathrm{D^b}(A) \simeq \mathrm{D^b}(B) \Longleftrightarrow \mathrm{D^b}(\mathrm{Kum}(A)) \simeq \mathrm{D^b}(\mathrm{Kum}(B)) \Longleftrightarrow \mathrm{Kum}(A) \simeq \mathrm{Kum}(B).$$

It is open what kind of statement one has in higher dimensions.

1.5 Appendix to Chapter 1

In this section we will give a different proof of Proposition 1.2.8. The strategy is as follows: First prove that any Mukai involution is an analytic morphism of the moduli space over $\mathbb{C}$, then show that it is in fact algebraic and using this conclude that a Mukai involution over $\overline{\mathbb{Q}}$ is a morphism as well.

Proposition 1.5.1. *Any Mukai involution is an automorphism of $\mathcal{M}^{\mathbb{C}}_{2d}$ considered as a complex-analytic space.*

Proof. Let $g_{\mathbb{C}}$ be an arbitrary Mukai involution sending X to $M_X(r, l, s) =: M(v)$. We know from [45] that there exists a Hodge isometry $\alpha : \Lambda \simeq v^\perp/\mathbb{Z}v$ (where $v^\perp$ is the orthogonal complement of v in the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$). It is clear that α induces an isomorphism $\mathbb{P}(\Lambda \otimes \mathbb{C}) \simeq \mathbb{P}((v^\perp/\mathbb{Z}v) \otimes \mathbb{C})$. This isomorphism restricts to an isomorphism of the period domains Ω and Ω' on both sides.

Now recall that if X is a K3 surface, then we get its associated period point $x_0 \in \Omega$ by choosing an isometry $\varphi : H^2(X, \mathbb{Z}) \simeq \Lambda$. We get the period point $\widetilde{x}_0$ of $M(v)$ in Ω by using the isometry $\alpha^{-1} \circ \psi^{-1} : H^2(M(v), \mathbb{Z}) \simeq \Lambda$ where $\psi : v^\perp/\mathbb{Z}v \xrightarrow{\simeq} H^2(M(v), \mathbb{Z})$ is the isometry from [45]. The map $g_{\mathbb{C}}$ sends x_0 to $\widetilde{x}_0$. Now, $\alpha(x_0)$ is by definition the period point of $M(v)$ considered as a point in Ω'. Thus, on the level of period domains the Mukai involution $g_{\mathbb{C}}$ is just the isomorphism α. Factoring out the markings is compatible with this process. The same proof works in the polarised setting. □

Proposition 1.5.2. *A Mukai involution $g_{\mathbb{C}}$ is an algebraic morphism.*

Proof. We want to apply the following theorem of Borel (see [12]):
If Y is a quasi-projective variety and $f : Y \longrightarrow \Omega/\Gamma$ is a holomorphic map to the quotient of a homogeneous symmetric domain Ω by an arithmetic torsion-free group Γ, then f is algebraic.

We wish to apply the theorem to $Y = \mathcal{M}_{2d}^{\mathbb{C}}$, $\Omega = \Omega_{2d}$, $\Gamma = \Gamma_{2d}$ and $f = g_{\mathbb{C}}$. Since Γ_{2d} is in general not torsion-free, we cannot apply the theorem directly. However, using results in [59] (compare also the proof of Prop. 2.2.2 in [28]) we can avoid this problem by using level covers and considering the algebraic construction of $\mathcal{M}_{2d}^{\mathbb{C}}$ described in Section 1.2. Denote the open subset of the Hilbert scheme used there by U.

For $l \geq 3$ consider $\Gamma_{2d}(l)$, the l-th congruence subgroup of Γ_{2d}. This group is torsion-free and the projection $\Omega_{2d}/\Gamma_{2d}(l) \longrightarrow \Omega_{2d}/\Gamma_{2d}$ is finite. Since the group of automorphisms of a polarised K3 surface that fix the polarisation is finite and this group acts faithfully on the cohomology of a K3 surface, we can apply [59, Prop. 2.17], which gives us a finite Galois covering U' of U such that $U' \longrightarrow U$ has Galois group $\Gamma_{2d}/\Gamma_{2d}(l)$. Thus we get a commutative diagram

$$\begin{array}{ccc} U'/PGL & \longrightarrow & \Omega_{2d}/\Gamma_{2d}(l) \\ \downarrow & & \downarrow \\ \mathcal{M}_{2d}^{\mathbb{C}} & \xrightarrow{g_{\mathbb{C}}} & \Omega_{2d}/\Gamma_{2d}. \end{array}$$

All the varieties in the diagram are quasi-projective and the vertical arrows are finite and surjective maps given as quotients of the action of a finite group. By Borel's result the upper map is algebraic and thus so is $g_{\mathbb{C}}$. □

To conclude one has to check that g is also a morphism. Since we are working over algebraically closed fields we can switch to the classical language and only consider closed points. So, g is a map of sets and by the above $g_{\mathbb{C}}$ is a morphism, i.e. locally on affine sets given by polynomials which a priori could have non-algebraic coefficients. However, we know that applied to points with $\overline{\mathbb{Q}}$-entries these polynomials give algebraic values.

Lemma 1.5.3. *Let $X \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$ be an affine variety with coordinate ring $K[X]$ and let $p \in K[X_{\mathbb{C}}]$ be a function having algebraic values on X. Then $p \in K[X]$.*

Proof. Set
$$H := \left\{ \sigma \in \mathrm{Aut}(\mathbb{C}) \ : \ \sigma_{|\overline{\mathbb{Q}}} = \mathrm{id} \right\},$$
then $\mathbb{C}^H := \{c \in \mathbb{C} \ : \ \sigma(c) = c \ \forall \, \sigma \in H\} = \overline{\mathbb{Q}}$. It follows that $\mathbb{C}[x_1, \ldots, x_n]^H = \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ and $K[X_{\mathbb{C}}]^H = K[X]$. Now consider $p - hp$ for a polynomial p as above and $h \in H$. By assumption we have that $p - hp$ is zero on $X_{\mathbb{C}}(\overline{\mathbb{Q}})$ and therefore also on $X_{\mathbb{C}}(\mathbb{C})$ since $X_{\mathbb{C}}(\overline{\mathbb{Q}}) \subset X_{\mathbb{C}}(\mathbb{C})$ is dense. Therefore $p - hp = 0 \in K[X_{\mathbb{C}}]$ and thus $p \in K[X]$. □

Thus we have re-proved the

Proposition 1.5.4. *The map g is a morphism.* □

Chapter 2

Stability conditions under change of base field

In this chapter we investigate the behaviour of Bridgeland stability conditions under change of base field, with particular focus on the case of a finite Galois extension. The main results are Theorem 2.2.22 which says that for a finite and separable extension L/K the stability manifold $\mathrm{Stab}(X)$ of a smooth projective variety X over a field K embeds as a closed submanifold into $\mathrm{Stab}(X_L)$ (where X_L is the base change scheme) via some naturally defined map, and Theorem 2.4.7 which states that if the numerical Grothendieck group does not change under field extension, then the stability manifold can only acquire new connected components over the larger field. Thus, if $\mathrm{Stab}(X)$ is non-empty and $\mathrm{Stab}(X_L)$ is connected, then these manifolds are homeomorphic.

We start by recalling the basic definitions of the theory of stability conditions in Section 2.1. One of the fundamental facts is that a stability condition can be viewed in two different ways: Via so called slicings or via hearts of bounded t-structures. Section 2.2 investigates base change from the point of view of slicings, whereas in Section 2.3 the latter point of view is used. Somewhat surprisingly, computations via hearts are fairly elusive and we can only prove some auxiliary results.

In the last section we work under the assumption that the Grothendieck group does not change under field extension, prove our second main result and apply it to the case of K3 surfaces.

2.1 Stability conditions

In this section we recall basic definitions and properties of Bridgeland's framework. Throughout we fix a K-linear essentially small triangulated category $\mathcal{T}$, which is furthermore assumed to be of finite type, that is, for any two objects E, F in $\mathcal{T}$ the K-vector space $\oplus_i \mathrm{Hom}_{\mathcal{T}}(E, F[i])$ is finite-dimensional over K.

Definition 2.1.1. A *stability condition* $\sigma = (Z, \mathcal{P})$ on $\mathcal{T}$ consists of a group homomorphism $Z : K(\mathcal{T}) \longrightarrow \mathbb{C}$, where $K(\mathcal{T})$ is the Grothendieck group of $\mathcal{T}$, and a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{T}$ for $\phi \in \mathbb{R}$, satisfying the following conditions:

(i) If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) > 0$.
(ii) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$ for all ϕ.
(iii) For $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$ we have $\mathrm{Hom}(E_1, E_2) = 0$.
(iv) For any $0 \neq E \in \mathcal{T}$ there exist finitely many real numbers $\phi_1 > \ldots > \phi_n$ and a collection of triangles $E_{i-1} \longrightarrow E_i \longrightarrow A_i$, $i \in \{1, \ldots n\}$, with $E_0 = 0$, $E_n = E$ and $A_i \in \mathcal{P}(\phi_i)$.

Recall that there is a bilinear form on $K(\mathcal{T})$, known as the *Euler form*, which is defined by the formula

$$\chi(E, F) = \sum_i (-1)^i \dim_K \mathrm{Hom}(E, F[i]).$$

A stability condition is *numerical*, if Z factors over the numerical Grothendieck group $N(\mathcal{T})$, which by definition is the quotient of $K(\mathcal{T})$ by the nullspace of the Euler form.

The map Z is called the *central charge*, the collection $\mathcal{P}$ *the slicing* and an object in $\mathcal{P}(\phi)$ *semistable of phase* ϕ. Given any interval $I \subset \mathbb{R}$ one defines $\mathcal{P}(I)$ to be the extension-closed subcategory of $\mathcal{T}$ generated by $\mathcal{P}(\phi)$ for $\phi \in I$. Recall that a subcategory $\mathcal{D}$ in $\mathcal{T}$ is called extension-closed, if the following condition if fulfilled: Whenever $A \longrightarrow B \longrightarrow C$ is a triangle in $\mathcal{T}$ and A, C are in $\mathcal{D}$, then B is also in $\mathcal{D}$.

Using this definition we can describe a connection to an important notion in the theory of triangulated categories, namely that of a t-structure, first introduced in [5]. Recall the

Definition 2.1.2. A *t-structure* on a triangulated category $\mathcal{T}$ is a full subcategory $\mathcal{F} \subset \mathcal{T}$, such that $\mathcal{F}[1] \subset \mathcal{F}$ and with the property that if one defines

$$\mathcal{F}^{\perp} = \{E \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(F, E) = 0 \text{ for all } F \in \mathcal{F}\},$$

then for every object $T \in \mathcal{T}$ there exists a triangle $F \longrightarrow T \longrightarrow E$ with $F \in \mathcal{F}$ and $E \in \mathcal{F}^{\perp}$.

A t-structure is called *bounded* if $\mathcal{T} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^{\perp}[j]$.

Note that the above categories $\mathcal{F}$ and $\mathcal{F}^{\perp}$ are in fact additive. This is easy to prove and will be used later on.

One often writes a t-structure as a pair of subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying the following conditions (i) and (ii). We use the notation $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$ for any $n \in \mathbb{Z}$.
(i) $\mathrm{Hom}(X, Y) = 0$ for every $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 1}$.
(ii) Every object $T \in \mathcal{T}$ fits into a triangle $\tau^{\leq 0}T \longrightarrow T \longrightarrow \tau^{\geq 1}T$ with $\tau^{\leq 0}T \in \mathcal{T}^{\leq 0}$ and $\tau^{\geq 1}T \in \mathcal{T}^{\geq 1}$.

The connection with the definition is given by the identification $\mathcal{T}^{\leq 0} = \mathcal{F}$ and $\mathcal{T}^{\geq 1} = \mathcal{F}^{\perp}$. The category $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is abelian and called the *heart* of the t-structure. The short exact sequences in $\mathcal{A}$ are precisely the exact triangles in $\mathcal{T}$ all of whose vertices are objects of $\mathcal{A}$. There are *truncation functors* $\tau_{\leq 0}$ and $\tau_{\geq 1}$ which are adjoint to the natural inclusions $\mathcal{T}^{\leq 0} \longrightarrow \mathcal{T}$ resp. $\mathcal{T}^{\geq 1} \longrightarrow \mathcal{T}$. Using shifts one can also define the functors $\tau_{\leq n}$ and $\tau_{\geq n}$ for any $n \in \mathbb{Z}$. We then define the *cohomology objects* of an object T by $H^0(T) := \tau_{\leq 0}\tau_{\geq 0}(T)$ and $H^i(T) = H^0(T[i])$. The guiding example is the *standard t-structure* on the bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ of an abelian category $\mathcal{C}$, where one defines

$\mathcal{F}$ to consist of those complexes E whose cohomology objects $H^i(E)$ are zero for $i > 0$. The heart of this t-structure is then precisely $\mathcal{C}$ and the abstractly defined cohomology objects are the usual ones.

We have the following easy

Lemma 2.1.3. *If $\sigma = (Z, \mathcal{P})$ is a stability condition, then the category $\mathcal{P}(> t)$ defines a bounded t-structure on $\mathcal{T}$ for all $t \in \mathbb{R}$. Similarly, the category $\mathcal{P}(\geq t)$ also defines a bounded t-structure.*

Proof. Property (ii) in the definition of a stability condition gives $\mathcal{P}(> t)[1] \subset \mathcal{P}(> t)$, by (iii) the category $\mathcal{P}(> t)^{\perp}$ is precisely $\mathcal{P}(\leq t)$ and the required triangle is given by (iv). The proof of the second statement is the same. $\square$

The heart of the t-structure $\mathcal{P}(> t)$ is the category $\mathcal{P}(t, t+1]$. As a matter of convention, one defines the heart of the slicing $\mathcal{P}$ to be $\mathcal{P}(0, 1]$. Clearly, one could alternatively work with the abelian category $\mathcal{P}(t, t+1]$ for any $t \in \mathbb{R}$. One could equally well use hearts of the form $\mathcal{P}[t, t+1)$.

The collection of triangles in (iv) is the *Harder–Narasimhan filtration* (abbreviated HN-filtration) of E and the A_i are the semistable factors. The HN-filtration is unique up to a unique isomorphism. Given an object $E \in \mathcal{T}$ one defines $\phi_\sigma^+(E) = \phi_1$, $\phi_\sigma^-(E) = \phi_n$ and the *mass* of E to be the number $m_\sigma(E) = \sum_{i=1}^n |Z(A_i)|$. With these definitions one for example has that the subcategory $\mathcal{P}(a, b)$, for an interval (a, b), consists of the zero object in $\mathcal{T}$ together with those non-zero objects E satisfying $a < \phi^-(E) \leq \phi^+(E) < b$.

There is an equivalent way of giving a stability condition. To do this define a *stability function* on an abelian category $\mathcal{A}$ to be a group homomorphism $Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the space $H := \{r \exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}$. The *phase* of an object $E \in \mathcal{A}$ is then defined to be

$$\phi(E) = 1/\pi \arg(Z(E)) \in (0, 1].$$

The phase allows one to order objects of $\mathcal{A}$ and it is thus possible to define semistable objects and HN-filtrations: An object $E \in \mathcal{A}$ is said to be *semistable* if every subobject $0 \neq F \subset E$ satisfies $\phi(F) \leq \phi(E)$. Equivalently, an object is semistable if for any nonzero quotient $E \twoheadrightarrow G$ one has $\phi(E) \leq \phi(G)$. Also recall the

Definition 2.1.4. Let $Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$. A *Harder–Narasimhan filtration* of a nonzero object $E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that the factors $F_i = E_i/E_{i-1}$ are semistable objects of $\mathcal{A}$ and

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

This generalises the classical case, where $\mathcal{A}$ is the category of coherent sheaves on a smooth projective curve and the ordering is done with respect to the slope $\mu = \deg / \mathrm{rk}$.

A stability function Z is said to have the *Harder–Narasimhan property* if any object possesses a HN-filtration (historically this property was first proved in

[26] for the slope function on the category of vector bundles on a curve). In this case we will call Z a stability condition. There is the following handy criterion for checking the HN-property:

Proposition 2.1.5. [14, Prop. 2.4] *Suppose a stability function $Z : K(\mathcal{A}) \longrightarrow \mathbb{C}$ satisfies the chain conditions*
(a) there are no infinite sequences of subobjects in $\mathcal{A}$

$$\cdots \subset E_{i+1} \subset E_i \subset \cdots \subset E_2 \subset E_1$$

with $\phi(E_{i+1}) > \phi(E_i)$ for all i,
(b) there are no infinite quotients in $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots E_i \twoheadrightarrow E_{i+1} \twoheadrightarrow \cdots$$

with $\phi(E_i) > \phi(E_{i+1})$ for all i. Then Z has the Harder-Narasimhan property.

The connection between the two concepts is given by

Proposition 2.1.6. [14, Prop. 5.3] *To give a stability condition on a triangulated category is equivalent to giving the heart $\mathcal{A}$ of a bounded t-structure and a stability function with HN-property on $\mathcal{A}$.*

Proof. First, recall that $K(\mathcal{A}) = K(\mathcal{T})$. Given $\sigma = (Z, \mathcal{P})$ one sets $\mathcal{A} = \mathcal{P}(0,1]$ and checks that Z defines a stability condition on $\mathcal{A}$. In the other direction given $(Z, \mathcal{A})$ one defines $\mathcal{P}(\phi)$ for $\phi \in (0,1]$ to be the semistable objects of phase ϕ with respect to Z. □

Remark 2.1.7. Again note that one could equally well define a stability condition on an abelian category $\mathcal{A}$ by replacing the subspace $H \subset \mathbb{C}$ by a strict half-plane

$$H_\alpha := \{r \exp(i\pi\phi) \mid r > 0 \text{ and } \alpha < \phi \leq \alpha + 1\} \subset \mathbb{C}.$$

This is what Bridgeland calls a *skewed stability condition*. It will be important to us that one could also work with strict half-planes of the form

$$H'_\beta := \{r \exp(i\pi\phi) \mid r > 0 \text{ and } \beta \leq \phi < \beta + 1\} \subset \mathbb{C}.$$

Proposition 2.1.5 still applies and, clearly, Proposition 2.1.6 still holds.

In order to exclude fairly pathological examples one only considers stability conditions with an additional property:

Definition 2.1.8. A stability condition is *locally finite*, if there exists some $\epsilon > 0$ such that for all $\phi \in \mathbb{R}$ the category $\mathcal{P}(\phi - \epsilon, \phi + \epsilon)$ is of finite length, i.e. Artinian and Noetherian.

The same definition applies to the above presented point of view via hearts. Note that the definitions clearly coincide as long as $\mathcal{P}(\phi-\epsilon, \phi+\epsilon) \subset \mathcal{P}(0,1]$ (or more generally $\mathcal{P}(\phi-\epsilon, \phi+\epsilon) \subset \mathcal{P}(t, t+1]$). Assume now that we have to prove that e.g. $\mathcal{P}(1-\epsilon, 1+\epsilon)$ is of finite length. Any object E in this category by definition is an extension of some objects $A \in \mathcal{P}(1-\epsilon, 1]$ and $B \in \mathcal{P}(1, 1+\epsilon)$. Both these categories are of finite length, hence A and B are Noetherian and Artinian and therefore E is as well.

Convention: From here on all stability conditions will be locally finite. The set of locally finite stability conditions on $\mathcal{T}$ will be denoted by $\mathrm{Stab}(\mathcal{T})$.

Recall that a generalised metric is a distance function on a set satisfying the usual metric space axioms except that it need not be finite. Bridgeland introduces a generalised metric on $\mathrm{Stab}(\mathcal{T})$ ([14, Prop. 8.1]):

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{T}} \left\{ |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|, |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, \left| \log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)} \right| \right\}$$

and proves the

Theorem 2.1.9. [14, Prop. 1.2] *For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{T})$ there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{T}), \mathbb{C})$ with a well-defined linear topology and a local homeomorphism $\mathcal{Z} : \Sigma \longrightarrow V(\Sigma)$ sending a stability condition $(Z, \mathcal{P})$ to its central charge Z. A similar result holds if one considers* $\mathrm{Stab}_N(\mathcal{T})$*, the set of numerical stability conditions, i.e. one substitutes $K(\mathcal{T})$ by $N(\mathcal{T})$.*

In particular, if $K(\mathcal{T}) \otimes \mathbb{C}$ is finite-dimensional, then $\mathrm{Stab}(\mathcal{T})$ is a finite-dimensional manifold.

There are two groups acting on the stability manifold: The group of exact autoequivalences $\mathrm{Aut}(\mathcal{T})$ and $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, the universal cover of $\mathrm{GL}^+(2, \mathbb{R})$. The former acts from the left by isometries as follows: For $\sigma = (Z, \mathcal{P})$ and $\Phi \in \mathrm{Aut}(\mathcal{T})$ we set $\Phi(\sigma) = (Z \circ \Phi^{-1}, \mathcal{P}')$ with $\mathcal{P}'(t) = \Phi(\mathcal{P}(t))$.

For the second action first recall that $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ can be thought of as pairs (T, f), where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, and $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an orientation-preserving automorphism such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2/\mathbb{R}_{>0}$ are the same. Now for a $\sigma \in \mathrm{Stab}(\mathcal{T})$ and $(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ define a new stability condition $\sigma' = (Z', \mathcal{P}')$ by setting $Z' = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

Notation: If Y is a smooth projective variety and $\mathrm{D^b}(Y)$ its bounded derived category of coherent sheaves, we will write $\mathrm{Stab}(Y)$ for the manifold of numerical locally finite stability conditions $\mathrm{Stab}_N(\mathrm{D^b}(Y))$. The Grothendieck group $K(\mathrm{Coh}(Y)) = K(\mathrm{D^b}(Y))$ will be denoted by $K(Y)$ and the numerical Grothendieck group by $N(Y)$.

Convention: From here on, unless stated otherwise, we will only consider numerical stability conditions.

Let us now have a look at the examples of smooth projective varieties where the numerical stability manifold is at least partially known. Since usually all results in the literature are formulated for schemes over the complex numbers, it might be useful to see whether the arguments used generalise to other fields.

Example 2.1.10. In [52] Okada proves that the stability manifold of $\mathbb{P}^1_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^2$. The main ingredients of the proof are Grothendieck's theorem about locally free sheaves on $\mathbb{P}^1$, namely that any such sheaf is a direct sum of the line bundles $\mathcal{O}(a_i)$, and the fact that the derived category of representations of the Kronecker quiver is equivalent to $\mathrm{D^b}(\mathbb{P}^1)$. These results are valid over an arbitrary field and therefore the stability manifold of $\mathbb{P}^1_K$ is isomorphic to $\mathbb{C}^2$ for any field K.

Example 2.1.11. Let C be a smooth projective curve of genus $g \geq 1$. There is a stability condition σ on the heart of the standard t-structure $\mathrm{Coh}(C)$ defined by $Z(E) = -\deg(E) + i\mathrm{rk}(E)$, for $E \in \mathrm{Coh}(C)$. In [14] Bridgeland proved that for an elliptic curve $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ acts freely on $\mathrm{Stab}(C)$ and that

$$\mathrm{Stab}(C) = \sigma \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R}).$$

Later Macrì ([39]) showed that the statement also holds for any $g \geq 1$. The proofs use Serre duality, the fact that the numerical Grothendieck group of a curve is isomorphic to $\mathbb{Z}^2$ and, in the case $g > 1$, the following technical statement proved in [22]:

Lemma [22, Lem. 7.2]: If C is a smooth projective curve of genus $g \geq 1$ and $E \in \mathrm{Coh}(C)$ is included in a triangle

$$Y \longrightarrow E \longrightarrow X \longrightarrow Y[1]$$

with $\mathrm{Hom}^{\leq 0}(Y, X) = 0$, then $X, Y \in \mathrm{Coh}(C)$.

The proof of this lemma is purely homological and, therefore, valid over an arbitrary field. Hence, the stability manifold of a smooth projective curve of genus $g \geq 1$ is isomorphic to $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ over an arbitrary field.

Example 2.1.12. The case of varieties with complete exceptional collections, in particular projective spaces, was investigated by Macrì in [39], who was able to determine a connected component of the stability manifold. Inspection of the arguments shows that they are valid over an arbitrary field.

We see that in the above cases the description of the stability manifold does not depend on the ground field. An explanation for this phenomenon seems to be that in these examples the structure of the derived category and/or the numerical Grothendieck group is particularly simple. Without these features the situation becomes more complicated:

Example 2.1.13. Let X be a complex projective K3 surface. We will use notation from Chapter 1. By the Riemann–Roch Theorem the Mukai vector map identifies the numerical Grothendieck lattice $(N(\mathrm{D^b}(X)), -\chi)$ with the lattice

$$N(X) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \subset H^*(X, \mathbb{Z})$$

with the Mukai pairing as the bilinear form. The set

$$\Delta(X) = \{\delta \in N(X) \mid \langle \delta, \delta \rangle = -2\}$$

is called the *root lattice* and for an element $\delta \in \Delta(X)$ we denote by $\delta^\perp$ the orthogonal complement of δ in $N(X) \otimes \mathbb{C}$ with respect to the complexified Mukai pairing. We need some more notation: Write $\mathcal{P}^\pm \subset N(X) \otimes \mathbb{C}$ for the two connected components of the set of those vectors whose real and imaginary part span a positive-definite two-plane in $N(X) \otimes \mathbb{R}$. Furthermore, note that a stability condition $(Z, \mathcal{P})$ is numerical if the central charge takes the form

$$Z(E) = (\pi(\sigma), v(E)),$$

where $\pi(\sigma)$ is some vector in $N(X) \otimes \mathbb{C}$ and $v(E)$ is, as before, the Mukai vector.

The main result in [15] is

Theorem [15, Thm. 1.1]: There is a connected component $\mathrm{Stab}^\dagger(\mathrm{X}) \subset \mathrm{Stab}(\mathrm{X})$ *which is mapped by* π *onto the open subset*

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset N(X) \otimes \mathbb{C}$$

Moreover, the induced map $\pi : \mathrm{Stab}^\dagger(\mathrm{X}) \longrightarrow \mathcal{P}_0^+(\mathrm{X})$ *is a covering map, and the subgroup of autoequivalences* $\mathrm{Aut}(\mathrm{D^b}(X))$ *which act trivially on the cohomology and preserve the connected component* $\mathrm{Stab}^\dagger(\mathrm{X})$ *acts freely on* $\mathrm{Stab}^\dagger(\mathrm{X})$ *and is the group of deck transformations of* π.

If X is a complex abelian surface, one has the following

Theorem [15, Thm. 15.2]: There is a connected component $\mathrm{Stab}^\dagger(\mathrm{X}) \subset \mathrm{Stab}(\mathrm{X})$ *which is mapped by* π *onto the open subset* $\mathcal{P}^+(X)$. *Moreover, the induced map* $\pi : \mathrm{Stab}^\dagger(\mathrm{X}) \longrightarrow \mathcal{P}^+(\mathrm{X})$ *is the universal cover, and the group of deck transformations is generated by the double shift functor* $[2]$.

Note that the formulation of similar results over fields other than $\mathbb{C}$ needs certain adjustments. We will come back to this example later.

2.2 Base change via slicings

Consider a field extension L/K, a smooth projective variety X over K, the base change scheme X_L over L and the flat projection $p : X_L \longrightarrow X$ which yields the exact faithful functor $p^* : \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X_L)$. Given a stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X_L)$ one is tempted to define $p_*(\sigma) := \sigma' = (Z', \mathcal{P}')$ as

$$Z' = Z \circ p^*$$

$$\mathcal{P}'(\phi) = \big\{ E \in \mathrm{D^b}(X) \,|\, p^*(E) \in \mathcal{P}(\phi) \big\} \quad \forall\, \phi \in \mathbb{R}.$$

It is very easy to see that $p_*(\sigma)$ satisfies conditions (i)-(iii) of the definition of a stability condition. Unfortunately the Harder–Narasimhan property need not hold: Looking at the definition of $p_*(\sigma)$ we see that an object $E \in \mathrm{D^b}(X)$ has a HN-filtration with respect to $p_*(\sigma)$ if and only if the HN-filtration of $p^*(E)$ with respect to σ is defined over the smaller field. For a possible counter-example cf. Remark 2.2.6.

Thus, this definition does not give a stability condition for an arbitrary $\sigma \in \mathrm{Stab}(X_L)$. Therefore, in general there is no natural map $\mathrm{Stab}(X_L) \longrightarrow \mathrm{Stab}(X)$.

Definition 2.2.1. For a field extension L/K define $\mathrm{Stab}(X_L)_p$ to be the subset of stability conditions on $\mathrm{D^b}(X_L)$ having the property that $p_*(\sigma)$ admits HN-filtrations. Thus we have a map

$$p_* : \mathrm{Stab}(X_L)_p \longrightarrow \mathrm{Stab}(X).$$

Remark 2.2.2. The geometric analogue of this is the following: Let $\pi : Y \longrightarrow Z$ be a finite unramified covering of smooth projective varieties and consider μ-semistability of coherent sheaves on Z resp. on Y with respect to $\mathcal{O}_Z(1)$ resp. $\mathcal{O}_Y(1) := \pi^*(\mathcal{O}_Z(1))$. Then it is well-known that a coherent sheaf F on Z is μ-semistable if and only if $\pi^*(F)$ is μ-semistable, cf. [35, Lem. 3.2.2].

Lemma 2.2.3. *The map p_* is continuous and, more precisely, for any $\sigma, \tau \in \mathrm{Stab}(X_L)_p$ we have $d(p_*(\sigma), p_*(\tau)) \leq d(\sigma, \tau)$. Its domain of definition $\mathrm{Stab}(X_L)_p$ is a closed subset of $\mathrm{Stab}(X_L)$.*

Proof. The first assertion follows from the second one and we recall the proof given in [40, Lem. 2.9]: Consider a $\sigma \in \mathrm{Stab}(X_L)_p$ and the HN-filtration of $E \in \mathrm{D^b}(X)$ with respect to $p_*(\sigma) = \sigma'$. The image of the filtration via the functor p^* is precisely the HN-filtration of $p^*(E)$ with respect to σ. Therefore, there are equalities $\phi^+_{\sigma'}(E) = \phi^+_\sigma(p^*E))$, $\phi^-_{\sigma'}(E) = \phi^-_\sigma(p^*(E))$ and $m_{\sigma'}(E) = m_\sigma(p^*(E))$. Consequently, setting $\tau' = p_*(\tau)$ and writing $\widetilde{E}$ for $p^*(E)$ one has

$$\begin{aligned}
d(\sigma', \tau') &= \sup_{0 \neq E \in \mathrm{D^b}(X)} \left\{ |\phi^+_{\sigma'}(E) - \phi^+_{\tau'}(E)|, |\phi^-_{\sigma'}(E) - \phi^-_{\tau'}(E)|, \left| \log \frac{m_{\sigma'}(E)}{m_{\tau'}(E)} \right| \right\} \\
&= \sup_{0 \neq E \in \mathrm{D^b}(X)} \left\{ |\phi^+_{\sigma}(\widetilde{E}) - \phi^+_{\tau}(\widetilde{E})|, |\phi^-_{\sigma}(\widetilde{E}) - \phi^-_{\tau}(\widetilde{E})|, \left| \log \frac{m_{\sigma}(\widetilde{E})}{m_{\tau}(\widetilde{E})} \right| \right\} \\
&\leq \sup_{0 \neq F \in \mathrm{D^b}(X_L)} \left\{ |\phi^+_{\sigma}(F) - \phi^+_{\tau}(F)|, |\phi^-_{\sigma}(F) - \phi^-_{\tau}(F)|, \left| \log \frac{m_{\sigma}(F)}{m_{\tau}(F)} \right| \right\} \\
&= d(\sigma, \tau)
\end{aligned}$$

Thus, p_* is continuous. The second assertion follows immediately from this computation. For the proof of the last assertion see [40, Lem. 2.8]. $\square$

Remark 2.2.4. The continuity of p_* also follows from the local description of the topology on the stability manifold (cf. Theorem 2.1.9) and the fact that the map $\mathrm{Hom}(N(X_L), \mathbb{C}) \longrightarrow \mathrm{Hom}(N(X), \mathbb{C})$ induced by pullback is continuous.

Remark 2.2.5. The reader will note that we did not address the question of local finiteness in the proof. In fact, it is automatic and we present the explanation given in [40, Rem. 2.7 (ii)]): The pullback functor induces a functor $\mathcal{P}'(\phi - \epsilon, \phi + \epsilon) \longrightarrow \mathcal{P}(\phi - \epsilon, \phi + \epsilon)$, which maps strict short exact sequences to strict short exact sequences. Let $i : E \longrightarrow F$ be an inclusion of objects $E, F \in \mathcal{P}'(\phi - \epsilon, \phi + \epsilon)$ and assume that $p^*(i) : p^*(E) \longrightarrow p^*(F)$ is an isomorphism. Then i is an isomorphism, because if i were not surjective, then there would exist a cokernel C which would then be mapped to zero by p^*. This is clearly impossible. Using this it is easy to check that $\mathcal{P}'(\phi - \epsilon, \phi + \epsilon)$ is of finite length, provided $\mathcal{P}(\phi - \epsilon, \phi + \epsilon)$ is.

Remark 2.2.6. Consider a heart $\mathcal{A}$ of a bounded t-structure $\mathcal{D}^{\leq 0}$ on $\mathrm{D^b}(X_L)$ which is of finite length and such that $\mathcal{D}^{\leq 0}$ does not descend to a t-structure on $\mathrm{D^b}(X)$, i.e.

$$\mathcal{C}^{\leq 0} = \left\{ E \in \mathrm{D^b}(X) \mid p^*(E) \in \mathcal{D}^{\leq 0} \right\}$$

is not a t-structure on $\mathrm{D^b}(X)$. We can define a stability condition on $\mathcal{A}$ by e.g. sending all simple objects to i. Thus, $\mathcal{P}(1/2) = \mathcal{A}$ and $\mathcal{P}(\phi) = 0$ for all $1/2 \neq \phi \in (0, 1]$. The HN-filtration of an object $p^*(E)$ in this example is nothing than the filtration of $p^*(E)$ with respect to the cohomology functors defined by $\mathcal{A}$. Since by assumption $\mathcal{A}$ does not descend, there exists an object $E_0 \in \mathrm{D^b}(X)$ such that the HN-filtration of $p^*(E_0)$ is not defined over the smaller field. Hence in general the subset $\mathrm{Stab}(X_L)_p$ will not be equal to $\mathrm{Stab}(X_L)$.

Lemma 2.2.7. *The map p_* is $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-equivariant and thus $\mathrm{Stab}(X_L)_p$ is. In particular, for $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}(X_L)_p$ the stability condition $\widetilde{\sigma}:=(rZ,\mathcal{P})$ is in $\mathrm{Stab}(X_L)_p$ for any $r\in\mathbb{R}_{>0}$.*

Proof. Consider a stability condition $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}(X_L)_p$, $\sigma'=(Z',\mathcal{P}')=p_*(\sigma)$ and an element $(T,f)\in\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. The stability condition $(T,f)\cdot(p_*(\sigma))=\sigma''=(Z'',\mathcal{P}'')$ is given by

$$Z''=T^{-1}\circ Z'=T^{-1}\circ Z\circ p^*$$

and

$$\mathcal{P}''(\phi)=\mathcal{P}'(f(\phi))=\{E\mid p^*(E)\in\mathcal{P}(f(\phi))\}\,.$$

On the other hand, $\widetilde{\sigma}=(T,f)\cdot\sigma$ is defined by $\widetilde{Z}=T^{-1}\circ Z$ and $\widetilde{\mathcal{P}}(\phi)=\mathcal{P}(f(\phi))$. It is now clear that $p_*(\widetilde{\sigma})$ is equal to σ'' and the lemma is therefore proved. □

Obviously one would like to to be able to say something about the structure of $\mathrm{Stab}(X_L)_p$, e.g. whether this set is always non-empty or connected. To tackle these questions we will further assume that the field extension is finite. We then also have the exact functor $p_*:\mathrm{D^b}(X_L)\longrightarrow\mathrm{D^b}(X)$ at our disposal.

In this situation we have $p_*(\mathcal{O}_{X_L})=\mathcal{O}_X^d$, where $d=[L:K]$. An immediate consequence of this is that for any stability condition $\sigma'=(Z',\mathcal{P}')$ on $\mathrm{D^b}(X)$ one has

$$p_*(\mathcal{O}_{X_L})\otimes\mathcal{P}'(\phi)=\mathcal{O}_X^d\otimes\mathcal{P}'(\phi)\subset\mathcal{P}'(\phi)\subset\mathcal{P}'[\phi,+\infty)$$

since the categories $\mathcal{P}'(\phi)$ are additive (in fact abelian). Since p is flat, it is trivially of finite Tor dimension and therefore we can apply [58, Cor. 2.2.2]. Note that in the statement of Cor. 2.2.2 the t-structure $\mathcal{P}'(>t)$ is used, whereas we use the t-structure $\mathcal{P}'(\geq t)$ (which is actually what is needed for the example following Cor. 2.2.2 in [58]). Thus, one has the

Proposition 2.2.8. *For a finite field extension L/K the map $p:X_L\longrightarrow X$ defines a continuous map*

$$p^*:\mathrm{Stab}(X)\longrightarrow\mathrm{Stab}(X_L).$$

Here, for a $\sigma'=(Z',\mathcal{P}')\in\mathrm{Stab}(X)$ we define $p^*(\sigma'):=\sigma=(Z,\mathcal{P})$ by $Z=Z'\circ p_*$ and $\mathcal{P}(\phi)=\left\{F\in\mathrm{D^b}(X_L)\mid p_*(F)\in\mathcal{P}'(\phi)\right\}$ for any $\phi\in\mathbb{R}$.

Proof. The only thing left to check is the continuity which is proved along the same lines as in Proposition 2.2.3. □

Remark 2.2.9. Similarly to p_* the map p^* satisfies $d(p^*(\sigma'),p^*(\tau'))\leq d(\sigma',\tau')$ for all $\sigma',\tau'\in\mathrm{Stab}(X)$. Also note that the local finiteness is again automatic, cf. Remark 2.2.5.

Remark 2.2.10. Once again, there is a geometric analogue of the above: With the assumptions of Remark 2.2.2 a coherent sheaf F' on Y is μ-semistable if and only $\pi_*(F')$ is μ-semistable, cf. [70, Prop. 1.5].

Recall that the group of automorphisms of X_L acts on $\mathrm{Stab}(X_L)$. In particular, we have an action of $G:=\mathrm{Aut}(L/K)$ on the stability manifold. Of course, these autoequivalences are not L-linear but only K-linear, but this is not relevant in our setting. Note that $\mathrm{D^b}(X_L)$ is of finite type as a K-linear category.

We can describe the image of p^* by formulating the

Proposition 2.2.11. *Let L/K be a finite extension and let σ' be an element of* $\mathrm{Stab}(X)$. *Then $p^*(\sigma')$ is invariant under the action of the group* $G = \mathrm{Aut}(L/K)$.

Proof. If $g \in G$ and $\sigma = p_*(\sigma')$, then $g(\sigma) = (Z \circ (g^{-1})^*, g^*(\mathcal{P}))$. Firstly, for $E \in \mathcal{P}(\phi)$ one has $g^*(p_*(E)) = p_*(g^*(E)) = p_*(E)$. Secondly, for any $E \in \mathrm{D^b}(X_L)$ the following holds:

$$Z(g^{-1})^*(E) = Z'p_*((g^{-1})^*(E)) = Z'(g^{-1})^*(p_*(E)) = Z'(p_*(E)) = Z(E)$$

We conclude that $g^*(\sigma) = \sigma$ as claimed. $\square$

Remark 2.2.12. Clearly the statement of the lemma is only interesting in the case when G is non-trivial, e.g. for a finite Galois extension.

Remark 2.2.13. In the case of a finite Galois extension one has $p_*p^*(E) = E^{\oplus d}$ for any $E \in \mathrm{D^b}(X)$ and $p^*p_* = \sum_{g \in G} g^*$, cf. e.g. [61].

Lemma 2.2.14. *If L/K is finite and Galois, then the composition $p_* \circ p^*$:* $\mathrm{Stab}(X) \longrightarrow \mathrm{Stab}(X)$ *is equal to the action of $h := (\frac{1}{d} \cdot \mathrm{id}, 1) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.*

Proof. Consider a stability condition $\sigma' = (Z', \mathcal{P}')$ in $\mathrm{Stab}(X)$. Then $p_*p^*(\sigma') = \sigma'' = (Z'', \mathcal{P}'')$ is defined as

$$Z'' = Z \circ p^* = Z' \circ p_* \circ p^* = dZ'$$

$$\mathcal{P}''(\phi) = \left\{F \in \mathrm{D^b}(X_L) \,|\, p_*p^*(F) = F^{\oplus d} \in \mathcal{P}'(\phi)\right\}$$

Clearly $\mathcal{P}'(\phi) \subset \mathcal{P}''(\phi)$. Let us prove the other inclusion. Assume $F \in \mathcal{P}''(\phi)$. Since σ' is a stability condition, F has a HN-filtration given by certain triangles $F_{i-1} \longrightarrow F_i \longrightarrow A_i$, $i \in \{1, \ldots, n\}$. Since the direct sum of triangles is a triangle, we can take the d-fold direct sum of these and get a filtration of $F^{\oplus d}$. But HN-filtrations are unique and by assumption $F^{\oplus d} \in \mathcal{P}'(\phi)$, so its HN-filtration is just

$$0 \longrightarrow F^{\oplus d} \xrightarrow{\mathrm{id}} F^{\oplus d}$$

Therefore $n = 1$, $\phi_1 = \phi$ and $F \in \mathcal{P}'(\phi)$. Thus, $p_* \circ p^*((Z', \mathcal{P}')) = (dZ', \mathcal{P}')$ as claimed. $\square$

Corollary 2.2.15. *For a finite Galois extension, the map $p_* \circ p^*$ is a homeomorphism, thus p^* is injective and p_* surjective.* $\square$

We will now investigate the domain of definition for the morphism p_*. To do this we need the following

Lemma 2.2.16. *If G is a finite group acting on a variety Y over a field K of characteristic prime to the order of the group, then any linearised object in* $\mathrm{D^b}(Y)$ *is isomorphic as a complex to a complex of G-linearised sheaves (cf. [16] and [57]).*

Proof. First recall that a linearised object is a pair (E, λ), where $E \in \mathrm{D^b}(Y)$ and λ is a collection of isomorphisms $\lambda_g : E \longrightarrow g^*(E)$ satisfying the usual cocycle condition, and morphisms between two such pairs are morphisms in $\mathrm{D^b}(Y)$ which are compatible with the linearisations. Denote the category of linearised objects by $\mathcal{T}$ and write $D^G(Y) = \mathrm{D^b}(\mathrm{Coh^G}(Y))$ for the bounded derived category of the abelian category $\mathrm{Coh^G}(Y)$ of linearised coherent sheaves on Y. Clearly, there is a functor $\Phi : D^G(Y) \longrightarrow \mathcal{T}$, which was proved to be an equivalence in [57]. We will use that it is fully faithful, but to make the dissertation more self-contained we will show the statement of the lemma (which is weaker than essential surjectivity) by induction on the number of cohomology objects. The case $n = 1$ is obvious. Let E be a complex with n cohomology objects. We may assume $H^i(E) = 0$ for $i \geq 2$. Consider the triangle given by the standard t-structure on $\mathrm{D^b}(Y)$:

$$\tau^{\leq 0}(E) =: E' \longrightarrow E \longrightarrow \tau^{\geq 1}(E) = H^1(E)[-1] \longrightarrow \tau^{\leq 0}(E)[1].$$

For any λ_g we get, since truncation is a functor, corresponding morphisms on E' and $H^1(E)$ and these morphisms define linearisations of these complexes. By induction $E' \simeq \Phi(F)$ and $H^1(E)[-1] \simeq \Phi(G)$ in $\mathcal{T}$ for some complexes $F, G \in D^G(Y)$. Since the morphisms in the triangle and the isomorphisms are compatible with the linearisations, the map $G[-1] \longrightarrow F$ is a map in $D^G(Y)$ and hence a cone is in $D^G(Y)$. This cone is isomorphic to E in $\mathrm{D^b}(Y)$. This concludes the proof. $\square$

Remark 2.2.17. We needed the assumption on the characteristic for the application of the result from [57] (the reason being that taking invariants is not well-behaved if the characteristic divides the order of the group).

Note that this proof does not show Φ to be essentially surjective, since the isomorphism does not need to respect the linearisations of E respectively the cone.

One could try to generalise the above statement to arbitrary faithfully flat morphisms replacing linearisations of sheaves resp. complexes by descent data. Note that the fully faithfulness used above is fulfilled in this situation.

Convention: From here on we assume that the characteristic of the ground field is prime to the order of the Galois group.

We also need the following

Lemma 2.2.18. *Let L/K be finite and Galois and consider $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X_L)_p$. Then $p^* \circ p_*(\sigma) = (\widetilde{Z}, \widetilde{\mathcal{P}})$, where*

$$\widetilde{Z}(E) = \sum_{g \in G} Z(g^*(E)) \quad \textit{and}$$

$$\widetilde{\mathcal{P}}(\phi) = \{E \mid \oplus_{g \in G}\, g^*(E) \in \mathcal{P}(\phi)\} = \left\{E \mid \{g^*(E)\}_{g \in G} \subset \mathcal{P}(\phi)\right\}.$$

Proof. Using Remark 2.2.13 we immediately get the formula for $\widetilde{Z}$ and the first equality for $\widetilde{\mathcal{P}}(\phi)$. As to the second equality: "$\supset$" holds because the categories $\mathcal{P}(\phi)$ are abelian, so in particular additive, and "$\subset$" holds because the categories $\mathcal{P}(\phi)$ are closed under direct summands. Let us prove this last statement: Let $A \oplus B$ be an element in $\mathcal{P}(\phi)$. We have to show that A and B are in $\mathcal{P}(\phi)$. Since $\mathcal{P}(\phi)$ is an abelian category, the kernel of the morphism

$$A \oplus B \xrightarrow{\begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}} A \oplus B,$$

which is of course B, is an element of $\mathcal{P}(\phi)$. Similarly one shows that $A \in \mathcal{P}(\phi)$. $\square$

We can now prove the

Proposition 2.2.19. *For a finite Galois extension L/K with Galois group G a stability condition $\sigma = (Z, \mathcal{P})$ is in $\mathrm{Stab}(X_L)_p$ if and only if its slicing $\mathcal{P}$ is invariant under the action of the Galois group, i.e. $E \in \mathcal{P}(\phi)$ implies $g^*(E) \in \mathcal{P}(\phi)$ for all $g \in G$. In particular, the subset of G-invariant stability conditions is contained in $\mathrm{Stab}(X_L)_p$.*

Proof. If $p_*(\sigma)$ is a stability condition on $\mathrm{D^b}(X)$, then we can consider $\widetilde{\sigma} = p^* \circ p_*(\sigma) \in \mathrm{Stab}(X_L)$. Lemma 2.2.18 shows that the slicing $\widetilde{\mathcal{P}}$ of $\widetilde{\sigma}$ is invariant under the action of the Galois group and that $\widetilde{\mathcal{P}}(\phi) \subset \mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$. In fact, we have an equality. To see this, consider $\phi \in \mathbb{R}$ and $E \in \mathcal{P}(\phi)$. We know that E has a HN-filtration with respect to $\widetilde{\sigma}$, i.e. it can be filtered by objects in $\widetilde{\mathcal{P}} \subset \mathcal{P}$. Since E is semistable with respect to σ, this filtration has to be trivial. Thus, $E \in \widetilde{\mathcal{P}}(\phi)$.

For the converse implication assume the slicing $\mathcal{P}$ of σ to be G-invariant. Consider the HN-filtration of an object $p^*(E)$, $E \in \mathrm{D^b}(X)$. Applying an arbitrary element $g \in G$ yields the HN-filtration of $g^*p^*(E) = p^*(E)$ with respect to σ, because the slicing is G-invariant. It follows that all the objects of the filtration are linearised objects of $\mathrm{D^b}(X_L)$. By Lemma 2.2.16 any such object is isomorphic to a complex of G-equivariant sheaves on X_L. Using Galois descent we see that a complex of equivariant objects is defined over K, and hence the HN-filtration is defined over K. $\square$

Remark 2.2.20. The stability function of a stability condition $\sigma \in \mathrm{Stab}(X_L)_p$ need not be G-invariant, since we only assume that the phase is constant on the orbits of semistable objects under the action of G. For an unstable object E the numbers $Z(E)$ and $Z(g^*(E))$ ($g \in G$) will in general not even be multiples of each other.

Using Lemma 2.2.18 we see that the restriction of $p^* \circ p_*$ to the subset of G-invariant stability conditions

$$\mathrm{Stab}(X_L)^G = \{\sigma \in \mathrm{Stab}(X_L) \mid g\sigma = \sigma \ \forall g \in G\}$$

is equal to the action of $h = (\frac{1}{d} \cdot \mathrm{id}, 1) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. In particular, the map $p^* : \mathrm{Stab}(X) \longrightarrow \mathrm{Stab}(X_L)^G$ is surjective. Therefore p^* is a homeomorphism, since we already have seen that it is injective. Thus, we have the following diagram

$$\begin{tikzcd}
\mathrm{Stab}(X_L)_p \arrow[dr, "p_*"'] & \mathrm{Stab}(X_L)^G \arrow[l, hook'] \\
& \mathrm{Stab}(X). \arrow[u, "p^*", "\simeq"']
\end{tikzcd}$$

We also need the following

Lemma 2.2.21. [40, Lem. 2.15] *The set of Galois-invariant stability conditions $\mathrm{Stab}(X_L)^G$ is a closed submanifold of $\mathrm{Stab}(X_L)$.*

Proof. Since in [40] the result is formulated for the action of a finite group on a smooth complex projective variety, we will give the first part of the proof to demonstrate that the arguments carry over to our case without difficulties: The subset $\mathrm{Stab}(X_L)^G$ is closed, because

$$\mathrm{Stab}(X_L)^G = \bigcap_{g \in G} (g, \mathrm{id})^{-1}(\Delta),$$

where Δ is the diagonal in $\mathrm{Stab}(X_L) \times \mathrm{Stab}(X_L)$ and every $g \in G$ acts continuously. The rest of the proof also works without changes. □

Using the lemma and the above discussion we immediately derive

Theorem 2.2.22. *For any finite and separable field extension L/K the map p^* realises $\mathrm{Stab}(X)$ as a closed submanifold of $\mathrm{Stab}(X_L)$.*

Proof. The statement is clear for a finite Galois extension from the discussion above and the previous lemma. The general case follows by considering the tower $K \subset L \subset L^n$, where L^n denotes the normal closure, the induced commutative diagram

$$\begin{tikzcd}
\mathrm{Stab}(X) \arrow[rr, "(p'')^*"] \arrow[dr, "p^*"'] & & \mathrm{Stab}(X_{L^n}) \\
& \mathrm{Stab}(X_L) \arrow[ur, "(p')^*"'] &
\end{tikzcd}$$

(where $p : X_L \longrightarrow X$, $p' : X_{L^n} \longrightarrow X_L$ and $p'' : X_{L^n} \longrightarrow X$ are the projections) and the fact that $L \longrightarrow L^n$ is Galois. □

Corollary 2.2.23. *Let L/K be a finite and separable extension and L^n the normal closure of L. Consider the commutative diagram*

$$\begin{tikzcd}
\mathrm{Stab}(X_{L^n}) \supseteq \mathrm{Stab}(X_{L^n})_{p''} \arrow[rr, "(p'')_*"] \arrow[dr, "(p')_*"] & & \mathrm{Stab}(X) \\
& \mathrm{Stab}(X_L) \supseteq \mathrm{Stab}(X_L)_p \arrow[ul, bend left, "(p')^*"] \arrow[ur, "p_*"'] &
\end{tikzcd}$$

(where the notation is as in the proof of Corollary 2.2.22). Then

$$\sigma \in \mathrm{Stab}(X_L)_p \Longleftrightarrow (p')^*(\sigma) \in \mathrm{Stab}(X_{L^n})_{p''}.$$

Proof. If $\sigma \in \mathrm{Stab}(X_L)_p$, then applying Lemma 2.2.14 to the pair $(p')_*$ and $(p')^*$ and using Lemma 2.2.7 we see that $(p')_*(p')^*(\sigma) \in \mathrm{Stab}(X_L)_p$ and hence $(p')^*(\sigma) \in \mathrm{Stab}(X_{L^n})_{p''}$. The converse is clear $(\mathrm{Stab}(X_{L^n})_{p''} \subset \mathrm{Stab}(X_{L^n})_{p'})$. □

Until the end of the section we will work in the Galois case. We know from Proposition 2.2.19 that the G-invariant stability conditions $\mathrm{Stab}(X_L)^G$ (which we identified with $\mathrm{Stab}(X)$) are contained in the set $\mathrm{Stab}(X_L)_p$. The next two results establish geometric connections between the two sets.

Lemma 2.2.24. *The subset $\mathrm{Stab}(X_L)^G$ is a retract of $\mathrm{Stab}(X_L)_p$.*

Proof. Recall that if $i : S \subset T$ is a pair of topological spaces, then by definition S is called a retract of T if there exists a map $f : T \longrightarrow S$ such that $f \circ i = \mathrm{id}_S$.

Define a map $f : \mathrm{Stab}(X_L)_p \longrightarrow \mathrm{Stab}(X_L)^G$ as follows: For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X_L)_p$ we set $f(\sigma) := \sigma' = (Z', \mathcal{P}')$, where $Z'(E) = (1/d) \sum_{g \in G} Z(g^*(E))$ and $\mathcal{P}' = \mathcal{P}$. It is fairly easy to show that σ' is a stability condition: Since by Proposition 2.2.19 the slicing of σ is G-invariant, the HN-filtrations of σ and σ' coincide. Furthermore, any object in $\mathcal{P}(\phi)$ clearly still has phase ϕ, since $Z(g^*(E))$ is of phase ϕ, for all $g \in G$.

Next, one has to verify that f is continuous: Let σ and τ be two stability conditions in $\mathrm{Stab}(X_L)_p$ such that $d(\sigma, \tau) < \delta$. Recall the definition of the generalised metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathrm{D^b}(X_L)} \left\{ |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|, |\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, \left| \log \frac{m_{\sigma_1}(E)}{m_{\sigma_2}(E)} \right| \right\}$$

and note that since the HN-filtrations do not change, neither do the first two numbers in the expression, i.e. the first two numbers are the same for the pairs (σ, τ) and $(f(\sigma), f(\tau))$. As for the last one note that for any semistable object A and any $g \in G$ one has $Z(g^*(A)) = \lambda_g Z(A)$ for some $\lambda_g > 0$ and hence $|Z(A) + Z(g^*(A))| = |Z(A)| + |Z(g^*(A))|$. Therefore

$$m_{f(\sigma)}(E) = \frac{1}{d} \left(\sum_{g \in G} m_\sigma(g^*(E)) \right)$$

for an arbitrary object E and hence the supremum over all E of $|\log \frac{m_{f(\sigma)}(E)}{m_{f(\tau)}(E)}|$ is small if the same holds for $|\log \frac{m_\sigma(E)}{m_\tau(E)}|$. We have thus proved that f is continuous.

Denoting the embedding of $\mathrm{Stab}(X_L)^G$ into $\mathrm{Stab}(X_L)_p$ by i we immediately conclude that the composition $f \circ i : \mathrm{Stab}(X_L)^G \hookrightarrow \mathrm{Stab}(X_L)_p \longrightarrow \mathrm{Stab}(X_L)^G$ is equal to the identity map and we therefore have a retraction. $\square$

In fact, a stronger statement is true:

Proposition 2.2.25. *The inclusion* $\mathrm{Stab}(X_L)^G \hookrightarrow \mathrm{Stab}(X_L)_p$ *is a deformation retract.*

Proof. Recall that $S \subset T$ is a deformation retract if there exists a homotopy $H : T \times [0,1] \longrightarrow T$ such that $H(-,0) = \mathrm{id}_T$, $H(-,1) \subset S$ and $H(s,1) = s$ for any $s \in S$. For details see [19].

The strategy of the proof is the following. Consider an element $\sigma = (Z, \mathcal{P})$ in $\mathrm{Stab}(X_L)_p$. Since $\mathcal{P}$ is already G-invariant, it seems natural to only deform Z until it also becomes G-invariant.

Assume for simplicity that $d = 2$ and consider the map

$$H : \mathrm{Stab}(X_L)_p \times [0,1] \longrightarrow \mathrm{Stab}(X_L)_p$$

sending (σ, t) to $(\widetilde{Z}, \widetilde{\mathcal{P}})$, where $\widetilde{Z}(E) = Z(E) + tZ(g^*(E))$ and $\widetilde{\mathcal{P}} = \mathcal{P}$. Clearly $H(\sigma, 0) = \sigma$ and $H(\sigma, 1) \in \mathrm{Stab}(X_L)^G$. Of course, with this definition $H(-,1) \neq \mathrm{id}$ on $\mathrm{Stab}(X_L)^G$, but this small problem is easily solved: Since for a $(Z', \mathcal{P}') = \sigma' \in \mathrm{Stab}(X_L)^G$ we have the equality $H(\sigma', 1) = (2Z', \mathcal{P}')$, it is easy to write down a homotopy from $H(-,1)$ to the identity map on $\mathrm{Stab}(X_L)^G$. The more challenging issue is the continuity of H. Inspecting the proof of the previous lemma it is easy to see that $H(-,t) : \mathrm{Stab}(X_L)_p \longrightarrow \mathrm{Stab}(X_L)_p$ is continuous, for any $t \in [0,1]$. Now fix a σ in $\mathrm{Stab}(X_L)_p$ and consider for simplicity only the question of continuity in 0. Looking at the definition of the metric on $\mathrm{Stab}(X_L)$, we see that the first two factors are zero, because the HN-filtrations of $H(\sigma,0) = \sigma$ and $H(\sigma,\epsilon) =: \tau$ are the same. Therefore we only need to consider the last factor. Take an arbitrary semistable object $A \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$. We know that $g^*(A)$ is again semistable of the same phase. Now, $m_\sigma(A) = |Z(A)|$ and $m_\tau(A) = (1+\epsilon\cdot\lambda_A)|Z(A)|$, where $\lambda_A = |Z(g^*(A))|/|Z(A)|$. Thus the quotient $(m_\tau(A))/(m_\sigma(A))$ is $(1+\epsilon\cdot\lambda_A)$ and we need λ_A to be bounded, if the last factor in the metric is to remain small. Since the numerical Grothendieck group has finite rank and the linear operator $(Z\circ g^*)/Z$ on $N(X_L)\otimes\mathbb{C} \supset N(X_L)$ has bounded norm, the quotient $|Z(g^*(A))|/|Z(A)|$ is indeed bounded by some constant C. Thus, for an arbitrary object $E \in \mathrm{D^b}(X_L)$ we get

$$m_\tau(E)/m_\sigma(E) = \frac{\sum_i |Z(A_i)| + \epsilon\sum_i |Z(g^*(A_i)|}{\sum_i |Z(A_i)|} =$$

$$= 1 + \frac{\epsilon(|Z(g^*(A_1))| + \ldots + |Z(g^*(A_n))|}{|Z(A_1)| + \ldots + |Z(A_n)|)} \leq 1 + \epsilon C$$

and thus $|\log(m_\tau(E)/m_\sigma(E))|$ is small provided ϵ is small enough. We conclude that H is continuous. For $d > 2$ one writes $G = \{g_1,\ldots,g_n\}$ and divides the interval $[0,1]$ up accordingly, so that in the first segment one changes Z to $Z_1 := Z + Z\circ g_1$, in the second Z_1 to $Z_2 := Z_1 + Z\circ g_2$ and so on. ☐

Remark 2.2.26. The proofs of all results in this section with the exception of Proposition 2.2.25 work for non-numerical locally finite stability conditions.

2.3 Base change via hearts

Recall that a stability condition $\sigma = (Z,\mathcal{P})$ can also be viewed as a pair consisting of a heart $\mathcal{D}$ and a stability condition Z on it, cf. Proposition 2.1.6. Thus, the stability manifold of a triangulated category $\mathcal{T}$ is partitioned with respect to the hearts of bounded t-structures on $\mathcal{T}$. Given a heart $\mathcal{D}$ in $\mathrm{D^b}(X_L)$ we can try to understand whether a stability condition $\sigma = (\mathcal{D}, Z)$ descends to a stability condition on $\mathrm{D^b}(X)$. One could apply a result like this to e.g. prove that $\mathrm{Stab}(X)$ is non-empty. In fact, this is precisely what we will do in the case of K3 surfaces at the end of this chapter. We start with the following

Proposition 2.3.1. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact functor between abelian categories such that ker$(F) = \{c \in \mathcal{C} \,:\, F(c) \simeq 0\} = 0$. If Z is a stability condition on $\mathcal{D}$, then composition with F induces a stability condition Z' on $\mathcal{C}$.*

Proof. The composition $Z \circ [\,] \circ F$, where $[\,] : \mathcal{D} \longrightarrow K(\mathcal{D})$ sends an object to its class, is clearly an additive function from $\mathcal{C}$ to $\mathbb{C}$ and hence by the universal property of the Grothendieck group we get a group homomorphism $Z' : K(\mathcal{C}) \longrightarrow \mathbb{C}$. By Proposition 2.1.5 we have to check whether there exists an infinite chain of subobjects/subquotients with increasing phases. By definition of Z' we have $\phi(C) = \phi(F(C))$ for any $C \in \mathcal{C}$. Assume e.g. that there exists an infinite chain of subobjects with increasing phases in $\mathcal{C}$. Using the triviality of the kernel and the exactness of F one gets an infinite chain in $\mathcal{D}$ with the same property. Since this is not possible, Z' has the Harder–Narasimhan property and thus is a stability condition. $\square$

Remark 2.3.2. If $\mathcal{C}$ and $\mathcal{D}$ are hearts of bounded t-structures in some triangulated categories and F is the restriction of an exact functor preserving the Euler forms, then the above statement holds for numerical stability conditions.

Consider a bounded t-structure with heart $\mathcal{D}$ on $\mathrm{D^b}(X_L)$. It defines a t-structure with heart $\mathcal{C}$ on $\mathrm{D^b}(X)$ if for any object $E \in \mathrm{D^b}(X)$ the filtration with respect to cohomology objects given by $\mathcal{D}$

$$\begin{tikzcd}
0 = F_0 \arrow[rr] & & F_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & F_{n-1} \arrow[rr] & & F_n = p^*(E), \arrow[dl] \\
& B_1 \arrow[ul, dashed] & & & & B_{n-1} \arrow[ul, dashed] &
\end{tikzcd}$$

where $B_i \in \mathcal{D}[i]$ for all i, is defined over the smaller field. Equivalently, the t-structure $\mathcal{D}^{\leq 0}$ on $\mathrm{D^b}(X_L)$ descends if the full subcategory

$$\mathcal{C}^{\leq 0} = \left\{ E \in \mathrm{D^b}(X) \mid p^*(E) \in \mathcal{D}^{\leq 0} \right\}$$

defines a t-structure on $\mathrm{D^b}(X)$.

Assume that a t-structure on $\mathrm{D^b}(X_L)$ given by a heart $\mathcal{D}$ descends to a t-structure on $\mathrm{D^b}(X)$, so in particular the category

$$\mathcal{C} = \left\{ E \in \mathrm{D^b}(X) \mid p^*(E) \in \mathcal{D} \right\}$$

is abelian. Note that since p^* respects the Euler forms, we can work with ordinary or numerical stability conditions and will restrict to the latter class. Denote the set of numerical stability conditions on the abelian category $\mathcal{C}$ resp. $\mathcal{D}$ by $\mathrm{Stab}(\mathcal{C})$ resp. $\mathrm{Stab}(\mathcal{D})$.

Corollary 2.3.3. *There exists a morphism* $\alpha : \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathrm{Stab}(\mathcal{C})$.

Proof. Since short exact sequences in $\mathcal{C}$ are nothing but distinguished triangles in $\mathrm{D^b}(X)$, the pull-back functor $F := p^* : \mathcal{C} \longrightarrow \mathcal{D}$ is exact. Furthermore, F is faithful, since the stalk of an object $p^*(E)$ in a point $y \in X_L$ equals the stalk of E in $p(y)$ tensorized with L. $\square$

Note that even if a t-structure $\mathcal{D}^{\leq 0}$ descends to a t-structure on $\mathrm{D^b}(X)$, it is not necessarily true that $\mathcal{D}^{\leq 0}$ is $\mathrm{Aut}(L/K)$-invariant. A priori one only has that if

$$D_1 \longrightarrow p^*(E) \longrightarrow D_2 \longrightarrow D_1[1]$$

is the decomposition of $p^*(E)$, where $E \in \mathrm{D^b}(X)$, with respect to the t-structure, i.e. $D_1 \in \mathcal{D}^{\leq 0}$ and $D_2 \in \mathcal{D}^{\geq 1}$, then the objects D_1 and D_2 are $\mathrm{Aut}(L/K)$-invariant. We cannot conclude this for objects not appearing in the decomposition of some $p^*(E)$. Under additional assumptions we can say something about the descent of a t-structure:

Proposition 2.3.4. *Let L/K be a finite Galois extension and let $\mathcal{D}^{\leq 0}$ be a t-structure on $\mathrm{D^b}(X_L)$. Then the t-structure descends if and only if it is Galois-invariant, i.e. for any $g \in G$ and $E \in \mathcal{D}^{\leq 0}$ one has $g^*(E) \in \mathcal{D}^{\leq 0}$.*

Proof. Assume that $\mathcal{D}^{\leq 0}$ descends. First, note that for an element $E \in \mathcal{D}^{\leq 0}$ one has that $p_*(E) \in \mathcal{C}^{\leq 0}$. To see this consider the decomposition of $p_*(E)$ with respect to $\mathcal{C}^{\leq 0}$:

$$C_1 \longrightarrow p_*(E) \longrightarrow C_2.$$

Pulling back to X_L gives the decomposition of $p^*p_*(E)$ with respect to $\mathcal{D}^{\leq 0}$:

$$p^*(C_1) \longrightarrow p^*p_*(E) \longrightarrow p^*(C_2).$$

Since $p^*p_*(E) = \sum_{g \in G} g^*(E)$ we conclude that this sequence is the direct sum of the decompositions of the $g^*(E)$ (use the easy fact that $\mathcal{D}^{\leq 0}$ is an additive category), so, in particular, E is a direct summand of $p^*(C_1)$. But then $p_*(E)$ a direct summand of $p_*p^*(C_1) = C_1^{\oplus d}$ and hence $p_*(E) \in \mathcal{C}^{\leq 0}$ as claimed. We now proceed to the proof that $\mathcal{D}^{\leq 0}$ is Galois-invariant. Take E is above. By what we just proved $\sum_{g \in G} g^*(E) \in \mathcal{D}^{\leq 0}$. Considering decompositions of $g^*(E)$ and taking the direct sum of these, which gives the decomposition

$$\textstyle\sum_{g \in G} g^*(E) \longrightarrow \sum_{g \in G} g^*(E) \longrightarrow 0,$$

we then immediately conclude that $g^*(E) \in \mathcal{D}^{\leq 0}$ for any $g \in G$.

The other direction is equally easy: Take an element $F \in \mathrm{D^b}(X)$, pull it back to X_L and consider its decomposition with respect to $\mathcal{D}^{\leq 0}$. Applying elements $g \in G$ to this triangle and using the invariance of $\mathcal{D}^{\leq 0}$ gives, via Galois descent, that the triangle is defined over K and hence the t-structure descends. $\square$

Having established this easy result, we now return to our discussion. The morphism α a priori does not correspond to the morphism p_* of the previous section. Of course, the stability function Z' is defined in the same way and the pullback of a semistable object E of phase ϕ in $\mathcal{C}$ by definition has the same phase in $\mathcal{D}$, but $p^*(E)$ is not necessarily semistable. Thus, the following definition is reasonable.

Definition 2.3.5. For a field extension L/K define $\mathrm{Stab}(X_L)_\alpha$ to be the set of stability conditions $\sigma = (\mathcal{D}, Z) \in \mathrm{Stab}(X_L)$ such that $\mathcal{D}$ descends to a heart in $\mathrm{D^b}(X)$ (i.e. the corresponding t-structure $\mathcal{D}^{\leq 0}$ descends).

Remark 2.3.6. In contrast to $\mathrm{Stab}(X_L)_p$ it seems difficult to show that the set $\mathrm{Stab}(X_L)_\alpha$ is closed in $\mathrm{Stab}(X_L)$ or that $\alpha : \mathrm{Stab}(X_L)_\alpha \longrightarrow \mathrm{Stab}(X)$ is continuous, the problem being that the pullback of the HN-filtration of an object $E \in \mathrm{D^b}(X)$ is not necessarily the HN-filtration of $p^*(E)$. What one can say is that $\mathrm{Stab}(X_L)_\alpha$ is preserved by the action of $\mathrm{Aut}(L/K)$. One further property of $\mathrm{Stab}(X_L)_\alpha$ is described in Proposition 2.3.9.

Under some additional assumptions we can establish a connection between α and p_*.

Proposition 2.3.7. *Let L/K be a finite Galois extension and let $\sigma = (\mathcal{D}, Z)$ be a stability condition on $\mathrm{D^b}(X_L)$ such that for any $\phi \in (0, 1]$ the subcategory* *of semistable objects in $\mathcal{D}$ of phase ϕ is invariant under the action of the Galois group (cf. Proposition 2.2.19), so, in particular, the heart $\mathcal{D}$ descends to $\mathrm{D^b}(X)$. Then $\alpha(Z)$ corresponds to $p_*(\sigma) = \sigma'$. We therefore have an inclusion $\mathrm{Stab}(X_L)_p \subset \mathrm{Stab}(X_L)_\alpha$ and $\alpha_{|\mathrm{Stab}(X_L)_p} = p_*$.*

Proof. We only have to show that for a semistable object $E \in \mathcal{C}$ of phase $\phi \in (0,1]$ the object $p^*(E)$ is again semistable (the converse is clear). Assuming the converse there exists a semistable object $F \subset p^*(E)$ such that $\phi_\sigma(F) > \phi_\sigma(p^*(E))$. Now, the functor p_* is exact and sends any object $X \in \mathcal{D}$ to an object in $\mathcal{C}$, since $\mathcal{C} = \left\{Y \in \mathrm{D^b}(X) \mid p^*(Y) \in \mathcal{D}\right\}$ and $p^*(p_*(X)) = \oplus_{g\in G} g^*(X) \in \mathcal{D}$. Thus, we can apply p_* to the inclusion $F \subset p^*(E)$ and get an inclusion $p_*(F) \subset p_*p^*(E) = E^{\oplus d}$. Since E is semistable in $\mathcal{C}$, so is $E^{\oplus d}$ and the phases are equal. But $p_*(F)$ is a destabilizing object of $E^{\oplus d}$ since

$$\begin{aligned}\phi_{\sigma'}(p_*(F)) &= \phi_\sigma(p^*p_*(F)) = \phi_\sigma(\oplus_{g\in G} g^*(F)) = \\ &= \phi_\sigma(F) > \phi_\sigma(p^*(E)) = \phi_{\sigma'}(E) = \phi_{\sigma'}(E^{\oplus d}).\end{aligned}$$

This is a contradiction, therefore an object E is semistable if and only if $p^*(E)$ is semistable. □

We can say a little bit more about descent of hearts using the theory of tilting. To do this first recall that a *torsion pair* in an abelian category $\mathcal{A}$ consists of two full additive subcategories $(\mathcal{T}, \mathcal{F})$ such that for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$ we have $\mathrm{Hom}(T, F) = 0$ and furthermore for any object $A \in \mathcal{A}$ there exists an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Note that the exact sequence is unique up to isomorphism.

The importance of torsion pairs is visible in the following

Proposition 2.3.8. [25, Prop. 2.1] *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in the heart $\mathcal{A}$ of a bounded t-structure on a triangulated category $\mathcal{T}$. Then the full subcategory*

$$\mathcal{A}^\sharp = \left\{E \in \mathcal{T} \mid H^i(E) = 0 \, \forall \, i \notin \{-1, 0\}, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}\right\}$$

is the heart of a bounded t-structure on $\mathcal{T}$. □

In the case of a finite Galois extension L/K we will say that $(\mathcal{T}, \mathcal{F})$ is a *Galois-invariant torsion pair*, if $g^*(T) \in \mathcal{T}$ and $g^*(F) \in \mathcal{F}$ for any $T \in \mathcal{T}$, $F \in \mathcal{F}$ and $g \in G$.

Proposition 2.3.9. *Assume that L/K is finite Galois, let $\sigma = (\mathcal{D}, Z)$ be a stability condition on $\mathrm{D^b}(X_L)$ such that its heart $\mathcal{D}$ descends to $\mathrm{D^b}(X)$ and assume that there is a Galois-invariant torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{D}$. Then the heart $\mathcal{D}^\sharp$, the tilt of $\mathcal{D}$ with respect to the torsion pair, also descends to $\mathrm{D^b}(X)$.*

Proof. We could simply invoke Proposition 2.3.4, but let us give a direct proof. Denote the descended heart defined by $\mathcal{D}$ by $\mathcal{C}$. It is fairly easy to see that the categories

$$\mathcal{T}' := \{E \in \mathcal{C} \mid p^*(E) \in \mathcal{T}\} \quad \text{and} \quad \mathcal{F}' := \{E \in \mathcal{C} \mid p^*(E) \in \mathcal{F}\}$$

define a torsion pair in $\mathcal{C}$: Clearly $\mathcal{T}'$ and $\mathcal{F}'$ fulfil the first requirement of a torsion pair. To see the second, consider an arbitrary object $C \in \mathcal{C}$ and the decomposition of its pullback with respect to the pair $(\mathcal{T}, \mathcal{F})$

$$0 \longrightarrow T \longrightarrow p^*(C) \longrightarrow F \longrightarrow 0.$$

Applying an arbitrary $g \in G$ to this sequence and using that the torsion pair is invariant, we conclude that the objects T and F are linearised and therefore $T \simeq p^*(T')$ and $F \simeq p^*(F')$ for some uniquely determined $T' \in \mathcal{T}'$ and $F' \in \mathcal{F}'$. Thus, $\mathcal{T}'$ and $\mathcal{F}'$ indeed define a torsion pair and this gives a new heart $\mathcal{C}^\sharp$. Thus, $\mathcal{D}^\sharp$ descends as claimed. □

Corollary 2.3.10. *The subset* $\mathrm{Stab}(X_L)_\alpha$ *is closed under tilting with respect to Galois-invariant torsion pairs.* □

Remark 2.3.11. Not every heart descends, even for a finite Galois extension and a structurally fairly simple derived category. Consider the extension $\mathbb{C}/\mathbb{R}$ and the derived category of $X = \mathbb{P}^1_{\mathbb{C}}$. Let $P \subset X$ be a non-empty set. We have a torsion pair in $\mathrm{Coh}(X)$ given by (see [22])

$$\mathcal{T} = \langle \mathcal{O}_x,\ x \in P \rangle \quad \text{and} \quad \mathcal{F} = \langle \mathcal{O}_y,\ y \notin P,\ \mathcal{O}(n), n \in \mathbb{Z} \rangle .$$

This gives a t-structure

$$\mathrm{D^b}(X)^{\leq 0} = \left\{ E \in \mathrm{D^b}(X)^{\leq 0} \mid H^0(E) \in \mathcal{T} \right\} .$$

Taking P to be a set which is not Galois-invariant, e.g. $P = [i : 0]$, we see that this t-structure does not descend to $\mathrm{D^b}(\mathbb{P}^1_{\mathbb{R}})$. Note however that there are no stability conditions on the heart of this t-structure (see [52]), but it seems plausible that in higher dimensions one can indeed find hearts which do not descend but have stability conditions on them.

2.4 Grothendieck groups

In this section we will return to the general case and consider non-numerical stability conditions as well. To maintain continuity we will keep the notation $\mathrm{Stab}(Y)$ for the numerical stability manifold of a variety Y and we will write $\mathrm{Stab}_*(Y)$ for the manifold of all locally finite stability conditions.

Theorem 2.1.9 tells us that the (numerical) stability manifold of a variety Y is locally homeomorphic to a subspace of $\mathrm{Hom}(K(Y), \mathbb{C})$ (resp. $\mathrm{Hom}(N(Y), \mathbb{C})$). Thus, it is natural to ask what happens with the (numerical) stability manifold under scalar extension if we have an isomorphism $K(X_L) \simeq K(X)$ (resp. $N(X_L) \simeq N(X)$).

The next proposition gives a first answer under a slightly weaker assumption:

Proposition 2.4.1. *Assume that L/K is a finite Galois extension and that the group homomorphism*

$$K(X) \otimes \mathbb{C} \longrightarrow K(X_L) \otimes \mathbb{C} \tag{2.4.1}$$

induced by p^* is an isomorphism. Then $\mathrm{Stab}_*(X_L)_p = \mathrm{Stab}_*(X_L)^G$. *Similarly, if we have an isomorphism for the numerical Grothendieck groups, then* $\mathrm{Stab}(X_L)_p = \mathrm{Stab}(X_L)^G$.

Proof. Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_*(X_L)_p$. We need to show that Z is constant on the orbits of the action of the Galois group G. Note that our assumption gives an isomorphism

$$\mathrm{Hom}(K(X), \mathbb{C}) \xrightarrow{\simeq} \mathrm{Hom}(K(X_L), \mathbb{C}).$$

We therefore can write $Z = Z' \circ p_*$ for some $Z' \in \mathrm{Hom}(K(X), \mathbb{C})$. For an object $E \in \mathrm{D^b}(X_L)$ and any $g \in G$ we then have

$$Z(g^*(E)) = Z' p_*(g^*(E)) = Z' g^*(p_*(E)) = Z' p_*(E) = Z(E)$$

as claimed. The proof in the numerical case is similar. □

Remark 2.4.2. Note that the homomorphism $p^* : N(X) \longrightarrow N(X_L)$ is always injective.

Remark 2.4.3. If (2.4.1) is an isomorphism, then we have the equalities

$$\mathrm{Stab}(X)_\alpha = \mathrm{Stab}(X)_p \quad \text{and} \quad \alpha = p_*.$$

To see this recall that we only need to check that if E is a semistable object in $\mathrm{D^b}(X)$ with respect to $\alpha(\sigma) = (\mathcal{C}, Z' = Z \circ p^*)$, then $p^*(E)$ is semistable with respect to $\sigma = (\mathcal{D}, Z)$. Assume for simplicity that E and $p^*(E)$ are in the corresponding hearts and let $F \subset p^*(E)$ be a destabilising object, i.e. $\phi_\sigma(F) > \phi_\sigma(p^*(E))$. By (2.4.1) we can write $Z = \widetilde{Z} \circ p_*$. Therefore, we have the equalities

$$Z'(p_*(F)) = Z \circ p^*(p_*(F)) = \widetilde{Z}(p_*(F)^{\oplus d}) = dZ(F)$$

and

$$Z'(p_* p^*(E)) = Z'(E^{\oplus d}) = dZ'(E) = dZ(p^*(E)).$$

Hence, $\phi_{\sigma'}(p_*(F)) > \phi_{\sigma'}(E)$, which is a contradiction.

Now recall from [14, Sec. 6] that for any $\sigma = (Z, \mathcal{P})$ in a connected component Σ of the stability manifold of a variety Y one can define a generalised norm on the vector space $\mathrm{Hom}(K(Y), \mathbb{C})$ by

$$\|U\|_\sigma = \sup \{ |U(E)|/|Z(E)| \ , \ E \text{ semistable in } \sigma \} \in [0, \infty]$$

In fact, the subspace $V(\Sigma)$ of Theorem 2.1.9 is the subspace of functions U for which the norm is finite. Bridgeland proves furthermore that if σ and τ are in the same connected component Σ, then the norms defined by these stability conditions are equivalent on $V(\Sigma)$.

Lemma 2.4.4. *Assume that (2.4.1) is an isomorphism and consider $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_*(X_L)_p$ and $p_*(\sigma) \in \mathrm{Stab}_*(X)$. Then the map*

$$\mathrm{Hom}(K(X), \mathbb{C}) \xrightarrow{\simeq} \mathrm{Hom}(K(X_L), \mathbb{C})$$

induced by p^* is continuous with respect to the topologies induced by σ and $p_*(\sigma)$. The same assertion holds for

$$\mathrm{Hom}(K(X_L), \mathbb{C}) \xrightarrow{\simeq} \mathrm{Hom}(K(X), \mathbb{C}),$$

where we consider the topologies induced by some $\sigma' \in \mathrm{Stab}_*(X)$ and $p^*(\sigma')$.

Proof. Let $V = U \circ p^* \in \mathrm{Hom}(K(X), \mathbb{C})$ and recall that $p_*(\sigma) = (Z \circ p^*, \mathcal{P}')$. Then

$$\|V\|_{p_*(\sigma)} = \sup\{|V(F)|/|Zp^*(F)|\ ,\ F \text{ semistable in } p_*(\sigma)\} =$$

$$\sup\{|Up^*(F)|/|Zp^*(F)|\ ,\ F \text{ semistable in } p_*(\sigma)\} \leq \|U\|_\sigma$$

since F by definition is $p_*(\sigma)$-semistable if $p^*(F)$ is σ-semistable. The proof for p_* is similar. $\square$

Corollary 2.4.5. *Assume $\sigma \in \mathrm{Stab}_*(X_L)_p$ is contained in some connected component Σ_L of dimension k and assume that (2.4.1) is an isomorphism. Then $p_*(\sigma)$ lies in a connected component Σ of dimension k. Similarly, the dimension remains the same under p^*. The same holds for numerical stability conditions.*

Proof. Follows immediately from the fact that k is the dimension of $V(\Sigma_L) = \{U\ ,\ \|U\|_\sigma < \infty\}$, the above computation and the fact that $p_*p^*(\sigma')$ respectively $p^*p_*(\sigma)$ are in the same connected component as σ resp. σ' by Lemma 2.2.14 resp. Lemma 2.2.18. The same proof works in the numerical case. $\square$

Proposition 2.4.6. *Let (2.4.1) be an isomorphism, let Σ be a component in $\mathrm{Stab}_*(X)$ and Σ_L the component of the same dimension in $\mathrm{Stab}_*(X_L)$ containing $p^*(\Sigma)$. Then we have $p^*(\Sigma) = \Sigma_L$. The same assertions hold for numerical stability conditions.*

Proof. Recall that we have an isomorphism between $V(\Sigma)$ and $V(\Sigma_L)$. Let $\sigma = (Z, \mathcal{P}) = p^*(\sigma') = p^*((Z', \mathcal{P}'))$ be an element in $p^*(\Sigma) \subset \Sigma_L$, $U \subset \Sigma_L$ be an open neighbourhood of σ homeomorphic to $U' \subset V(\Sigma_L)$ and V be an open neighbourhood of σ' homeomorphic to $V' \subset V(\Sigma)$. Restricting to the intersection $p_*(V') \cap U'$ if necessary and abusing notation we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "p^*"] \arrow[d, "\simeq"] & U \arrow[d, "\simeq"] \\ V(\Sigma) \arrow[r, "\simeq"] & V(\Sigma_L). \end{tikzcd}$$

This shows that $p^*(\Sigma) \subset \Sigma_L$ is open. Since it is also closed by Theorem 2.2.22 and Σ_L is connected, we have the claimed equality. The proof for numerical stability conditions is similar. $\square$

The proposition immediately implies

Theorem 2.4.7. *If $N(X) \otimes \mathbb{C} \simeq N(X_L) \otimes \mathbb{C}$, $\mathrm{Stab}(X_L)$ is connected and $\mathrm{Stab}(X)$ is non-empty, then $\mathrm{Stab}(X) \simeq \mathrm{Stab}(X_L)$. A similar statement holds for the manifolds of all locally finite stability conditions.* $\square$

We will now apply the above results to the case of K3 surfaces. Recall the notation and the statements of Example 2.1.13.

Assume that the K3 surface $S_\mathbb{C}$ is defined over the real numbers and possesses an $\mathbb{R}$-rational point. Further assume that all line bundles on $S_\mathbb{C}$ are also defined over $\mathbb{R}$. This is e.g. the case if $S_\mathbb{C}$ is generic, i.e. of Picard rank 1, since in this case the Galois group has to act as the identity on $\mathbb{Z} = \mathrm{Pic}(S_\mathbb{C})$. These conditions ensure that (2.4.1) (or rather its numerical version) is an isomorphism.

It is well-known that there cannot exist any numerical stability conditions on the standard heart $\mathrm{Coh}(S_{\mathbb{C}})$. Bridgeland therefore uses the theory of tilting to produce new hearts on which stability conditions can indeed be constructed. The method works as follows: One takes $\mathbb{R}$-divisors β and ω so that ω is in the ample cone

$$\mathrm{Amp}(S_{\mathbb{C}}) = \left\{\omega \in \mathrm{NS}(S_{\mathbb{C}}) \otimes \mathbb{R} \mid \omega^2 > 0 \text{ and } \omega \cdot C > 0 \text{ for any curve } C \subset S\right\}.$$

Recall that the slope $\mu_\omega(E)$ of a torsion-free sheaf E on $S_{\mathbb{C}}$ with respect to ω is defined by

$$\mu_\omega(E) = \frac{c_1(E) \cdot \omega}{\mathrm{rk}(E)}.$$

This gives us the possibility to define semistability with respect to the slope. It turns out that any torsion-free sheaf has a HN-filtration with respect to μ_ω. One can then show that there exists a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathrm{Coh}(S_{\mathbb{C}})$ defined as follows: The category $\mathcal{T}$ consists of those sheaves whose torsion-free parts have μ_ω-semistable HN-factors of slope $\mu_\omega > \beta \cdot \omega$ and $\mathcal{F}$ consists of torsion-free sheaves on S all of whose μ_ω-semistable HN-factors have slope $\mu_\omega \leq \beta \cdot \omega$.

It follows from this construction that the torsion pair does not depend on β, but only on ω and the product $\beta \cdot \omega$. If ω is an ample line bundle (and therefore by our assumption defined over $\mathbb{R}$), then the HN-filtration of a sheaf pulled back from S is defined over $\mathbb{R}$, cf. [35, Thm. 1.3.7]. The torsion pair therefore descends and so does the heart obtained by tilting with respect to it. Corollary 2.3.3 then implies that the stability manifold $\mathrm{Stab}(S)$ is non-empty. Ignoring other possible components and using the corollary above we thus see that the distinguished component described by Bridgeland is defined over $\mathbb{R}$. We thus proved the

Proposition 2.4.8. *Let S be a K3 surface over $\mathbb{R}$ and denote by $S_{\mathbb{C}}$ the complex K3 surface obtained by base change. Furthermore, assume that S has an $\mathbb{R}$-rational point and that $\mathrm{Pic}(S) = \mathrm{Pic}(S_{\mathbb{C}})$. Then there exists a connected component $\mathrm{Stab}^\dagger(\mathrm{S}) \subset \mathrm{Stab}(\mathrm{S})$ such that there is a homeomorphism between $\mathrm{Stab}^\dagger(\mathrm{S})$ and Bridgeland's distinguished component $\mathrm{Stab}^\dagger(\mathrm{S}_{\mathbb{C}}) \subset \mathrm{Stab}(\mathrm{S}_{\mathbb{C}})$.* $\square$

Chapter 3

Scalar extensions for triangulated categories

In this chapter we propose a construction which associates an L-linear triangulated category to a K-linear triangulated category and a field extension L/K. We use the notion of L-modules: If C is an object in some arbitrary K-linear category $\mathcal{C}$, then an L-module structure is by definition a morphism of K-algebras from L to the endomorphisms of C in $\mathcal{C}$. This notion is well-known in the context of additive and abelian categories and we give a review in the first section. Since we believe that triangulated categories are not rigid enough for this construction to be applied, we take a detour via pretriangulated differential graded categories. The necessary facts and results are presented in the second section. In Section 3.3 we give our definition of scalar extension and prove our main results, Propositions 3.3.4, 3.3.5 and 3.3.7, which state that our construction produces the expected results in some standard examples. In the last section we study the behaviour of the dimension of a triangulated category under base change and prove that it does not vary if the extension is finite and Galois, for the precise statement see Propositions 3.4.4 and 3.4.5.

3.1 Scalar extensions for additive categories

Definition 3.1.1. Let $\mathcal{C}$ be a K-linear additive category and let L/K be a field extension. The *base change category* $\mathcal{C}_L$ is defined as follows:
• Objects of $\mathcal{C}_L$ are pairs (C, f), where $C \in \mathcal{C}$ and $f : L \longrightarrow \mathrm{End}_{\mathcal{C}}(C)$ is a morphism of K-algebras.
• Morphisms between (C, f) and (D, g) are given by morphisms $\alpha : C \longrightarrow D$ in $\mathcal{C}$ compatible with the given actions of L, i.e. for any $l \in L$ the diagram

$$\begin{tikzcd} C \arrow[r, "f(l)"] \arrow[d, "\alpha"] & C \arrow[d, "\alpha"] \\ D \arrow[r, "g(l)"] & D \end{tikzcd}$$

commutes.

We call the datum (C, f) an *L-module structure* on C.

Lemma 3.1.2. *The category $\mathcal{C}_L$ is additive and comes with a natural L-linear structure.*

Proof. The verification is straightforward: The zero object is $(0,0)$, the direct sum of (C,f) and (D,g) is given by $(C \oplus D, f \oplus g)$, the K-linearity is obvious. As to the L-linearity: For a scalar $l \in L$ and an $\alpha \in \mathrm{Hom}_{\mathcal{C}_L}((C,f),(D,g))$ define $l \cdot \alpha := \alpha \circ f(l) = g(l) \circ \alpha$. It is then easy to check that this is well-defined and thus $\mathcal{C}_L$ is indeed L-linear. $\square$

Lemma 3.1.3. *If $\mathcal{C}$ is an abelian category, then $\mathcal{C}_L$ is also abelian.*

Proof. Let $\alpha : (C,f) \longrightarrow (D,g)$ be a morphism in $\mathcal{C}_L$. We first have to show the existence of a kernel and a cokernel. We will show the existence of the former, the latter is similar. Forgetting the additional structures there exists a kernel A in $\mathcal{C}$. One can define a canonical morphism $h : L \longrightarrow \mathrm{End}_{\mathcal{C}}(A)$ as follows: Let $l \in L$ be arbitrary and consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \xrightarrow{\alpha} & D \\ & & \big\downarrow{\scriptstyle f(l)} & & \big\downarrow{\scriptstyle g(l)} \\ A & \xrightarrow{i} & C & \xrightarrow{\alpha} & D \end{array}$$

Since $\alpha \circ f(l) \circ i = g(l) \circ \alpha \circ i = 0$, there exists a unique morphism $A \xrightarrow{h(l)} A$ making the diagram commutative. This defines h and makes i a morphism in $\mathcal{C}_L$. The axiom about the equality of the image and the coimage is equally easy to check. $\square$

Let us now consider base change for functors:

Definition 3.1.4. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between K-linear abelian (or additive) categories. The functor $F_L : \mathcal{A}_L \longrightarrow \mathcal{B}_L$ is defined as follows: For an L-module $(A,f) \in \mathcal{A}_L$ define a module structure $\widetilde{f}$ on $F(A)$ by the composition $L \longrightarrow \mathrm{End}_{\mathcal{A}}(A) \longrightarrow \mathrm{End}_{\mathcal{B}}(F(A))$, i.e. $\widetilde{f}(l) = F(f(l))$. For any $\alpha : (A,f) \longrightarrow (A',g)$ the map $F(\alpha)$ is then compatible with the module structures on $F(A)$ and $F(A')$ and this defines F on morphisms.

Note that with this definition F_L is exact if F is. Furthermore one has

Lemma 3.1.5. *If F is an equivalence, then F_L is also an equivalence.*

Proof. Let (A,f) and (A',g) be objects in $\mathcal{A}_L$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}((A,f),(A',g)) & \xrightarrow{F_L} & \mathrm{Hom}((F(A),\widetilde{f}),(F(A'),\widetilde{g})) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}(A,A') & \xrightarrow[\simeq]{F} & \mathrm{Hom}(F(A),F(A')) \end{array}$$

where the vertical maps are the inclusions. The functor F_L is therefore faithful. Let $\beta : F(A) \longrightarrow F(A')$ be compatible with the module structures. Since F is full, there exists an α such that $F(\alpha) = \beta$ and we have

$$F(\alpha \circ f(l)) = F(\alpha) \circ F(f(l)) = F(\alpha) \circ \widetilde{f}(l) = \widetilde{g}(l) \circ F(\alpha) = F(g(l) \circ \alpha)$$

for any $l \in L$. Since F is faithful, this shows $\alpha \circ f(l) = g(l) \circ \alpha$. We conclude that α is a morphism in $\mathcal{A}_L$ and F_L is full.
Finally, let (B, h) be an element in $\mathcal{B}_L$. Using the inverse functor F^{-1} we get an object $(A, f) \in \mathcal{A}_L$ such that $F_L((A, f)) = (B, h)$ and thus F_L is essentially surjective. $\square$

Example 3.1.6. Let A be a K-algebra and let $\mathcal{C} = \mathrm{Mod}(A)$ be the abelian category of (left) A-modules. As one would expect, one has an equivalence

$$\Phi : \mathrm{Mod}(A)_L \simeq \mathrm{Mod}(A \otimes_K L).$$

The definition of the functor is straightforward: If (M, f) is an element in $\mathrm{Mod}(A)_L$, then one can define an $(A \otimes_K L)$-module structure on M as follows:

$$(A \otimes_K L) \times M \longrightarrow M, \qquad (a \otimes l, m) \longmapsto f(l)(am)$$

A morphism $\alpha : (M, f) \longrightarrow (N, g)$ is simply sent to itself, since the compatibility with the L-module structures translates into linearity over $A \otimes_K L$. It is obvious that Φ is faithful. It is full since for any $(A \otimes_K L)$-linear map $\varphi : M \longrightarrow N$ one has

$$\varphi f(l)(am) = \varphi((a \otimes l)m) = (a \otimes l)\varphi(m) = g(l)(a\varphi(m)) = g(l)\varphi(am)$$

and therefore φ can be considered as a map from (M, f) to (N, g). Finally, the functor is essentially surjective since any $(A \otimes_K L)$-module M can be considered as an A-module and the L-module structure is given by

$$L \longrightarrow \mathrm{End}_A(M), \qquad l \longmapsto \left[f(l) : m \longmapsto \mu(1 \otimes l)m\right],$$

where μ is the scalar multiplication.

Using similar arguments one also proves $\mathrm{QCoh}(X)_L \simeq \mathrm{QCoh}(X_L)$ for any scheme X over K.

Note that the same argument shows that for a finite field extension the base change of the abelian category of all finitely generated A-modules is equivalent to the category of all finitely generated $(A \otimes_K L)$-modules. It follows that for a noetherian scheme X over K one has an equivalence $\mathrm{Coh}(X)_L \simeq \mathrm{Coh}(X_L)$.

Remark 3.1.7. The group $\mathrm{Aut}(L/K)$ acts on $\mathcal{C}_L$ in the following way: Let $\alpha \in \mathrm{Aut}(L/K)$ and $(A, f) \in \mathcal{C}_L$, then $\alpha(A, f) := (A, f \circ \alpha)$. If $\mathcal{C}$ is equal to $\mathrm{Mod}(A)$ for a K-algebra A, then it is easy to see that this action corresponds to the usual action of $\mathrm{Aut}(L/K)$ on modules.

Lemma 3.1.8. *Let $\mathcal{C}$ be a K-linear additive category. Then we have an equivalence:*

$$F : Kom(\mathcal{C})_L \xrightarrow{\simeq} Kom(\mathcal{C}_L).$$

Proof. Let $(A^\bullet = \ldots \longrightarrow A^i \longrightarrow A^{i+1} \longrightarrow \ldots, f)$ be an object in $Kom(\mathcal{C})_L$ so that for any $l \in L$ one has a morphism of complexes $f(l) : A^\bullet \longrightarrow A^\bullet$. For any $n \in \mathbb{Z}$ the component $f(l)^n$ defines an L-module structure on A^n and the differentials are compatible with these structures, hence are morphisms in $\mathcal{C}_L$. Therefore, $A^\bullet \in Kom(\mathcal{C}_L)$ and F is defined on objects. A morphism $\alpha : (A^\bullet, f) \longrightarrow (B^\bullet, g)$ is simply sent to α considered as a morphism of complexes in $Kom(\mathcal{C}_L)$. It is now obvious that F is an equivalence. $\square$

There is a forgetful functor $\Lambda : \mathcal{C}_L \longrightarrow \mathcal{C}$ from the base change category to the original one, which is exact in the abelian case. It is also possible to define a functor in the other direction as follows:

If $\mathcal{C}$ is a K-linear additive category, V a finite-dimensional K-vector space and $X \in \mathcal{C}$ one can consider the functor

$$F_V^X : \mathcal{C} \longrightarrow Vec_K, \qquad C \longmapsto \mathrm{Hom}_K(V, \mathrm{Hom}_{\mathcal{C}}(X, C)).$$

This functor is representable by the object $X^{\oplus \dim_K(V)}$ which will, for obvious reasons, be denoted by $V \otimes_K X$. Here we tacitly assume that either the field extension is finite or that $\mathcal{C}$ has arbitrary direct sums. Using the defining property of $V \otimes_K X$ one has an isomorphism

$$\mu : F_V^X(V \otimes_K X) = \mathrm{Hom}_K(V, \mathrm{Hom}_{\mathcal{C}}(X, V \otimes_K X)) \simeq \mathrm{End}(V \otimes_K X).$$

Let us now specialize to $V = L$, where L is our finite field extension. We can define an L-module structure on $L \otimes_K X$ as follows. Consider the element $f_0 = \mu^{-1}(\mathrm{id}) \in \mathrm{Hom}_K(L, \mathrm{Hom}_{\mathcal{C}}(X, L \otimes_K X))$. Any element $l \in L$ gives a K-linear map from L to itself and therefore we can define $\alpha(l)$ to be $\mu(f_0 \circ l)$. It is easy to check that this defines a homomorphism of algebras $\alpha : L \longrightarrow \mathrm{End}(L \otimes_K X)$ and thus an L-module structure on $L \otimes_K X$. One could equally well just use the following

Lemma 3.1.9. *Let $\mathcal{A}$ be an additive category with arbitrary direct sums. There exist canonical maps* $\mathrm{Mat}(I \times J, K) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(\oplus_I X, \oplus_J X)$*, where I and J are some index sets, which are compatible with the inclusions and projections. Via these maps, matrix multiplication corresponds to composition of maps.*

Proof. This is a special case of [1, Lem. B3.3]. Note that in [1] the authors work with abelian categories, but the quoted lemma only needs the additivity. □

Mapping X to $X \otimes_K L$ defines an exact K-linear functor

$$\Xi : \mathcal{C} \longrightarrow \mathcal{C}_L$$

by sending an exact sequence $X \longrightarrow Y \longrightarrow Z$ to its d-fold sum. One has the

Lemma 3.1.10. *The functor Ξ is left adjoint to Λ, i.e. for objects $C \in \mathcal{C}$ and $(D, \alpha) \in \mathcal{C}_L$ one has a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}_L}(\Xi(C), (D, \alpha)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(C, \Lambda(D, \alpha)).$$

Proof. We recall the proof from [67] where an inverse is constructed as follows: Let f be an element in $\mathrm{Hom}_{\mathcal{C}}(C, D)$. Using α one defines a morphism $L \longrightarrow \mathrm{Hom}(C, D)$ by $l \longmapsto \alpha(l) \circ f$. By definition of the tensor product this corresponds to a morphism $\Xi(C) \longrightarrow D$ which is compatible with the L-module structures. One could also just quote [1, Prop. B3.16]. Note that in [1] the authors define tensor products in a more general setting and therefore abelian categories have to be used for some of the arguments. In our situation the additivity is in fact sufficient for the quoted statement. □

Example 3.1.11. Consider the situation of Example 3.1.6. It is easy to see that the functor Ξ corresponds to tensoring an A-module with the ring $A \otimes_K L$ and the functor Λ is nothing but considering a module over $A \otimes_K L$ as an A-module. Going from the affine situation to an arbitrary scheme X over K we see that Ξ corresponds to p^* and Λ to p_*, where $p : X_L \longrightarrow X$ is the projection. Of course, p^* is exact, since p is flat. Thus, the above lemma translates into the usual adjunction of the functors p^* and p_*.

Corollary 3.1.12. *If (C, f) is an injective object in $\mathcal{C}_L$, then $C = \Lambda(C, f)$ is an injective object in $\mathcal{C}$. Furthermore, if $\mathcal{C}_L$ has enough injective objects, then the same holds for $\mathcal{C}$.*

Proof. By the above lemma the functor $\mathrm{Hom}(-, C)$ is isomorphic to the functor $\mathrm{Hom}(\Xi(-), (C, f))$ which is exact, being the composition of the exact functors Ξ and $\mathrm{Hom}(-, (C, f))$. This proves the first statement. As to the second one: Consider an arbitrary element $C \in \mathcal{C}$. The object $\Xi(C)$ can, by assumption, be embedded into an injective object (D, g). Applying the exact functor Λ to this embedding we get an injection $C^{\oplus d} \longrightarrow D$. Thus, C can be embedded into the injective object D. $\square$

Remark 3.1.13. In fact, the converse implication of the second statement also holds, cf. [38, Prop. 4.8].

Corollary 3.1.14. *There is a fully faithful functor $I(\mathcal{C}_L) \longrightarrow I(\mathcal{C})_L$ sending (I, f) to (I, f) (where $I(\mathcal{C})$ resp. $I(\mathcal{C}_L)$ denotes the category of injective objects in $\mathcal{C}$ resp. $\mathcal{C}_L$). Furthermore, $I(\mathcal{C}_L)$ is closed under direct summands in $I(\mathcal{C})_L$.*

Proof. Only the second statement needs a proof. Let (I, f) and (J, g) be two elements in $I(\mathcal{C})_L$ such that their direct sum $(I \oplus J, f \oplus g)$ is in $I(\mathcal{C}_L)$. Now use that a direct summand of an injective object is injective. $\square$

3.2 Differential graded categories

Let us now discuss why such an approach cannot work for triangulated categories (of course, the main example we have in mind is the derived category of a scheme X over K). Let $\alpha : (X, f) \longrightarrow (Y, g)$ be an arbitrary morphism in a triangulated category $\mathcal{T}$ commuting with the given L-module structures on X and Y. One now has to define an L-module structure on $C(\alpha)$, a cone of α. It seems that the only way to do this would be the following: Let $l \in L$ and consider the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f(l)"] & Y \arrow[r, "i"] \arrow[d, "g(l)"] & C(\alpha) \arrow[r, "\pi"] & X[1] \arrow[d, "f(l)[1]"] \\ X \arrow[r, "\alpha"] & Y \arrow[r, "i"] & C(\alpha) \arrow[r, "\pi"] & X[1] \end{tikzcd}$$

By TR3 one has a mophism $h(l) : C(\alpha) \longrightarrow C(\alpha)$. However, this morphism is not unique and from the equalities $f(l'l) = f(l')f(l)$ and $g(l'l) = g(l')g(l)$ $(l, l' \in L)$ and the commutativity of the appropriate diagrams we only get

$$h(l'l)i = h(l')h(l)i \quad \text{and} \quad \pi h(l'l) = \pi h(l')h(l).$$

Hence, we cannot conclude that one gets an L-module structure on $C(\alpha)$ and it seems that the underlying problem is the non-uniqueness of the cone. Therefore, it seems natural to take a detour via enhanced triangulated categories. We start with differential graded categories (for details see e.g. [36]) by recalling the

Definition 3.2.1. A *differential graded category* or *DG-category* over a field K is a K-linear category $\mathcal{A}$ such that for any objects $X, Y \in \mathcal{A}$ the space of morphisms $\mathrm{Hom}(X, Y)$ is a complex and the composition of morphisms

$$\mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)$$

is a chain map.

Since for any $X \in \mathcal{A}$ the algebra $\mathrm{End}(X)$ is a DG-algebra (i.e. an algebra which has a structure of a complex such that the Leibniz rule holds for the differential), it seems reasonable to take this into account when defining an L-module structure in this setting. Regarding L as a DG-algebra over K with trivial differential this consideration leads to

Definition 3.2.2. If X is an object of a DG-category $\mathcal{A}$, then a stucture of an L-module on X is given by a morphism $f : L \longrightarrow \mathrm{End}_{\mathcal{A}}(X)$ of DG-algebras over K.

In particular, the image of L under f lies in the kernel of d^0 of $\mathrm{End}_{\mathcal{A}}(X)$.

We thus have a category $\mathcal{A}_L$ of L-modules. It is easy to prove the

Lemma 3.2.3. *For a DG-category $\mathcal{A}$ over K the category $\mathcal{A}_L$ has the structure of a DG-category over L.*

Proof. One only needs to check that the space of morphisms between two L-modules (X, f) and (Y, g) is a complex in a natural way. For this it is enough to show that for any $\alpha \in \mathrm{Hom}((X, f), (Y, g))$ the map $d(\alpha)$ is again in $\mathrm{Hom}((X, f), (Y, g))$, in other words that the differential in $\mathrm{Hom}(X, Y)$ restricts to the subgroup $\mathrm{Hom}((X, f), (Y, g))$.
We know that $\alpha f(l) = g(l)\alpha$ for any $l \in L$. Differentiating both sides gives

$$d(\alpha)f(l) + \alpha d(f(l)) = d(\alpha f(l)) = d(g(l)\alpha) = d(g(l))\alpha + g(l)d(\alpha).$$

Since f and g are morphisms of DG-algebras, $d(f(l)) = f(d(l)) = f(0) = 0$ and similarly for g. This completes the proof. $\square$

Convention: If $f : L \longrightarrow \mathrm{End}_{\mathcal{A}}(X)$ and $g : L \longrightarrow \mathrm{End}_{\mathcal{A}}(Y)$ are two given module structures, we will sometimes write $\mathrm{Hom}^{f,g}(X, Y)$ for the subcomplex $\mathrm{Hom}((X, f), (Y, g))$ of $\mathrm{Hom}(X, Y)$ defined above.

Example 3.2.4. The most basic example of a K-linear DG-category is the category of complexes of K-vector spaces. For two complexes X and Y we define $\mathrm{Hom}(X, Y)^n$ to be the K-vector space formed by families $\alpha = (\alpha^p)$ of morphisms $\alpha^p : X^p \longrightarrow Y^{p+n}$, $p \in \mathbb{Z}$. We define $\mathrm{Hom}_{DG}(X, Y)$ to be the graded K-vector space with components $\mathrm{Hom}(X, Y)^n$ and whose differential is given by

$$d(\alpha) = d_Y \circ \alpha - (-1)^n \alpha \circ d_X.$$

The DG-category $C_{DG}(K)$ has as objects complexes and the morphisms are defined by

$$C_{DG}(K)(X,Y) = \mathrm{Hom}_{DG}(X,Y).$$

Of course, starting with the category of complexes over an arbitrary K-linear abelian (or additive) category one can associate a DG-category to it in a similar manner.

Clearly, we get back the usual category of complexes by taking as morphisms only the closed morphisms of degree zero and we get the usual homotopy category if we replace $\mathrm{Hom}_{DG}(X,Y)$ by $\ker(d^0)/\operatorname{im}(d^{-1})$.

Let us show that the base change category $C_{DG}(K)_L$ is naturally equivalent to the category $C_{DG}(L)$. If (X,f) is an L-module, then the image of the map $f : L \longrightarrow \mathrm{End}_{DG}(X)$ is contained in $\ker(d^0)$, i.e. in the space of chain maps. Therefore, as in Lemma 3.1.8, we see that (X,f) gives an object in $C_{DG}(L)$ and we therefore have a functor $C_{DG}(K)_L \longrightarrow C_{DG}(L)$. The rest of the proof is clear.

Recall that a *DG-functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between DG-categories $\mathcal{A}$ and $\mathcal{B}$ is by definition required to be compatible with the structure of complexes on the spaces of morphisms.

Definition 3.2.5. Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be two DG-functors. We define *the complex of graded morphisms* $\mathrm{Hom}(F,G)$ to be the complex whose nth component is the space formed by families of morphisms $\phi_X \in \mathrm{Hom}_{\mathcal{B}}(F(X),G(X))^n$ such that $(G\alpha)(\phi_X) = (\phi_Y)(F\alpha)$ for all $\alpha \in \mathrm{Hom}_{\mathcal{A}}(X,Y)$, where $X, Y \in \mathcal{A}$. The differential is given by that of $\mathrm{Hom}_{\mathcal{B}}(F(X),G(X))$. Using this we define the DG-category of DG-functors from $\mathcal{A}$ to $\mathcal{B}$, denoted by $\mathrm{Hom}(\mathcal{A},\mathcal{B})$, to be the category with DG-functors as objects and the above described spaces as morphisms.

The next proposition provides a different description of the base change category.

Proposition 3.2.6. *Let 1_L be the K-linear DG-category with one object whose endomorphism ring is L. For a K-linear DG-category $\mathcal{A}$ define $\mathcal{A}'_L$ to be the category $\mathrm{Hom}(1_L,\mathcal{A})$. Then there exists an equivalence $\mathcal{A}'_L \simeq \mathcal{A}_L$.*

Proof. Let $F : 1_L \longrightarrow \mathcal{A}$ be a functor. It determines a unique object $X \in \mathcal{A}$. Furthermore, if F is a DG-functor, we get a homomorphism of DG-algebras $f : L \longrightarrow \mathrm{End}(X)$. Thus F corresponds to (X,f), an L-module. By definition the natural transformations between two functors F and G correspond precisely to morphisms from X to Y compatible with the module structures which finishes the proof. □

Remark 3.2.7. Let $\mathcal{A}$ be an additive DG-category and assume that either L/K is finite or that $\mathcal{A}$ has arbitrary direct sums. In this situation there exists a natural DG-functor $\mathcal{A} \longrightarrow \mathcal{A}_L$ defined as in Section 2. Using the above description it is given as the functor mapping $A \in \mathcal{A}$ to the functor sending the unique object of 1_L to $A^{\oplus \dim_K(L)}$.

Remark 3.2.8. Note that there is a second possibility to associate to a K-linear DG-category $\mathcal{A}$ an L-linear DG-category, namely by taking the tensor product of $\mathcal{A}$ with the category 1_L. Recall that the tensor product of two DG-categories

$\mathcal{A}$ and $\mathcal{B}$ is defined to be the DG-category where the objects are pairs (A, B) and the space of morphisms of two such pairs (A, B) and (A', B') is defined to be the tensor product of complexes $\mathrm{Hom}_{\mathcal{A}}(A, A') \otimes \mathrm{Hom}_{\mathcal{B}}(B, B')$. However, this cannot be the right construction in the geometric case, since we do not get any new objects. It rather seems that in a sense this construction corresponds to associating to $\mathrm{Coh}(X)$ (for a scheme X over K) the category $p^*(\mathrm{Coh}(X))$, where $p : X_L \longrightarrow X$ is the projection.

Recall that to any DG-category $\mathcal{A}$ one can naturally associate two other categories: Firstly, there is the *graded category* $Ho^\bullet(\mathcal{A})$ having the same objects as $\mathcal{A}$ and where the space of morphisms between two objects X, Y is by definition the direct sum of the cohomologies of the complex $\mathrm{Hom}_{\mathcal{A}}(X, Y)$. Secondly, restricting to the cohomology in degree zero we get the *homotopy category* $Ho(\mathcal{A})$. Recall further that a DG-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *quasi-equivalence* if for any two objects X, Y in $\mathcal{A}$ the map

$$\mathrm{Hom}(X, Y) \longrightarrow \mathrm{Hom}(F(X), F(Y))$$

is a quasi-isomorphism and furthermore the induced functor $Ho(F)$ is essentially surjective. A DG-functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a *DG-equivalence* if it is fully faithful and for every object $B \in \mathcal{B}$ there is a closed isomorphism of degree 0 between B and an object of $F(\mathcal{A})$. We also have to recall the following construction from [8].

Definition 3.2.9. Let $\mathcal{A}$ be a DG-category. Define the *pretriangulated hull* $\mathcal{A}^{pretr}$ of $\mathcal{A}$ to be the following category. Its objects are formal expressions $(\oplus_{i=1}^n C_i[r_i], q)$, where $C_i \in \mathcal{A}$, $r_i \in \mathbb{Z}$, $n \geq 0$, $q = (q_{ij})$, $q_{ij} \in \mathrm{Hom}(C_j, C_i)[r_i - r_j]$ is homogeneous of degree 1, $q_{ij} = 0$ for $i \geq j$, $dq + q^2 = 0$. If $C = (\oplus_{j=1}^n C_j[r_j], q)$ and $C' = (\oplus_{i=1}^m C'_i[r'_i], q)$ are objects in $\mathcal{A}^{pretr}$, then the $\mathbb{Z}$-graded K-module $\mathrm{Hom}(C, C')$ is the space of matrices $f = (f_{ij})$, $f_{ij} \in \mathrm{Hom}(C_j, C'_i)[r'_i - r_j]$ and the composition map is matrix multiplication. The differential $d : \mathrm{Hom}(C, C') \longrightarrow \mathrm{Hom}(C, C')$ is defined by $d(f) = (df_{ij}) + q'f - (-1)^l fq$ if $\deg f_{ij} = l$. The category $\mathcal{A}$ is called *pretriangulated* if the natural fully faithful functor $\Psi : \mathcal{A} \longrightarrow \mathcal{A}^{pretr}$ is a quasi-equivalence and $\mathcal{A}$ is *strongly pretriangulated* if Ψ is a DG-equivalence.

Remark 3.2.10. There is an equivalent way of describing the pretriangulated hull. To do this, recall that for any K-linear DG-category $\mathcal{A}$ the category of DG-functors $\mathrm{Hom}(\mathcal{A}, C_{DG}(K))$ is called the category of *DG-modules* and denoted by $\mathrm{Mod}(\mathcal{A})$. As in the classical setting there is a Yoneda embedding $\mathcal{A} \longrightarrow \mathrm{Mod}(\mathcal{A})$ and an element in the image is called *representable*. A DG-module Φ is called *semi-free* if there exists a filtration $0 = \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi$ such that Φ_{k+1}/Φ_k is isomorphic to a direct sum of shifts of representable modules. A semi-free DG-module is *finitely generated* if $\Phi_n = \Phi_{n+1}$ for all $n \gg 0$ and Φ_{k+1}/Φ_k is a finite direct sum. As explained in [8] there is a canonical embedding $\mathcal{A}^{pretr} \longrightarrow \mathrm{Mod}(\mathcal{A})$ and under this embedding $\mathcal{A}^{pretr}$ is DG-equivalent to the category of semi-free finitely generated DG-modules (cf. [20]).

3.3 Scalar extension of a triangulated category

Definition 3.3.1. Let $\mathcal{T}$ be a K-linear triangulated category and let L/K be a field extension. We define the *base change category* $\mathcal{T}_L$ as follows: Choose an enhancement of $\mathcal{T}$, i.e. an additive pretriangulated DG-category $\mathcal{A}$ such that $Ho(\mathcal{A}) \simeq \mathcal{T}$, and define $\mathcal{T}_L$ to be the smallest thick (i.e. closed under taking direct summands) full triangulated subcategory of $Ho((\mathcal{A}_L)^{pretr})$ containing the image of $\mathcal{T}$ under the functor induced by $\mathcal{A} \longrightarrow \mathcal{A}_L \hookrightarrow (\mathcal{A}_L)^{pretr}$.

Remark 3.3.2. In the above definition and in the following we tacitly assume that $\mathcal{A}$ has infinite direct sums or that L/K is finite.

Remark 3.3.3. Clearly one would like to see that this definition does not depend on the enhancement. Unfortunately we were not able to prove this statement so far. The difficulty is that the internal Hom-functor (which we use, cf. Proposition 3.2.6) in the 2-category of DG-categories does not respect quasi-equivalences. Therefore, in the following the notation should probably reflect that an enhancement has been chosen, but we will not do this.

Let us now consider the results this construction produces in some standard examples.

Proposition 3.3.4. *Let X be a smooth projective variety over K and consider $\mathrm{D^b}(X)$. Then $(\mathrm{D^b}(X))_L$ is equivalent to $\mathrm{D^b}(X_L)$.*

Proof. We know that $\mathcal{T} \simeq \widetilde{K}^+(I(\mathcal{C}))$, where $I(\mathcal{C})$ is the additive category of injective objects in $\mathcal{C} = \mathrm{QCoh}(X)$ and $\widetilde{K}^+(I(\mathcal{C}))$ is the homotopy category of bounded-below complexes of injectives having only finitely many coherent cohomology objects. It is well-known that the DG-category of bounded-below complexes of injective objects with bounded coherent cohomology $\mathcal{A} = C_{DG}(I(\mathcal{C}))$ is an enhancement of $\mathcal{T}$. Base change for this DG-category produces $C_{DG}(I(\mathcal{C})_L)$, which is a pretriangulated DG-category and therefore taking the pretriangulated hull does not change its homotopy category. Using Corollary 3.1.12 it is easy to see that $I(\mathcal{C}_L)$ can be embedded as a full thick subcategory into $I(\mathcal{C})_L$ and hence $\mathrm{D^b}(X_L) = \widetilde{K}^+(I(\mathcal{C}_L))$ is a full triangulated thick subcategory in $\widetilde{K}^+(I(\mathcal{C})_L) = Ho(\mathcal{A}_L)$ (where $\widetilde{K}^+$ is defined similarly as above). Clearly, $\mathrm{D^b}(X_L)$ contains $\mathrm{D^b}(X)$. In fact, $\mathrm{D^b}(X_L)$ is the smallest thick triangulated subcategory of $Ho(\mathcal{A}_L)$ with this property: In [55] it is shown that the category $\mathrm{D^b}(X_L)$ has a classical generator, i.e. an object E with the property that the smallest triangulated thick subcategory of $\mathrm{D^b}(X_L)$ containing E is everything. Now use that the classical generator E is a direct sum of tensor powers of the very ample line bundle and therefore is in the image of the functor $\mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X_L)$. □

There is a slight variation of the above:

Proposition 3.3.5. *If X is a noetherian scheme over K and L/K is a finite Galois extension, then $(\mathrm{D^b}(X))_L \simeq \mathrm{D^b}(X_L)$.*

Proof. As in the previous proposition one shows that $\mathrm{D^b}(X_L)$ contains $\mathrm{D^b}(X)$. To show that $\mathrm{D^b}(X_L)$ is indeed the smallest thick triangulated subcategory of $Ho(\mathcal{A}_L)$ (notation as before) one uses the formula

$$p^*p_*(E) = \sum_{g \in G} g^*(E),$$

where $p : X_L \longrightarrow X$ is the projection and G is the Galois group. $\square$

Remark 3.3.6. There exists a different enhancement of $\mathrm{D^b}(X)$ if X is smooth and projective. We will need some notation: Denote by $\mathcal{C}(X)$ the pretriangulated DG-category consisting of bounded-below complexes of $\mathcal{O}_X$-modules with bounded coherent cohomology. Now, we know that $\mathrm{D^b}(X)$ is equivalent to the category $\mathrm{Perf}(X)$ of perfect complexes, that is, finite complexes of vector bundles. Choosing a finite affine covering $\mathcal{U}$ of X, one has the (strongly) pretriangulated DG category $\mathcal{P}(\mathcal{U}) \subset \mathcal{C}(X)$ which, by definition, is the smallest full DG-subcategory of $\mathcal{C}(X)$ containing all Čech resolutions of elements of $\mathrm{Perf}(X)$ and closed under taking cones of closed morphisms of degree zero. This category is an enhancement of $\mathrm{D^b}(X)$ by [11, Lem. 6.7]. It is easy to see that the category $\mathcal{P}(\mathcal{U})_L$ is equivalent to $\mathcal{P}(\mathcal{U}_L)$ (where $\mathcal{U}_L$ is the affine covering of X_L given by pulling back $\mathcal{U}$) and hence its homotopy category is equivalent to $\mathrm{D^b}(X_L)$. Using the same arguments as in the above propositions one sees that our definition produces the expected result if one works with this enhancement.

We also have the following result in the non-geometric situation:

Proposition 3.3.7. *Let $\mathcal{C}$ be an abelian category with enough injectives and with generators (for details see e.g. [42, Ch. II, 15]). Then $\mathrm{D^b}(\mathcal{C})_L$ is equivalent to $\mathrm{D^b}(\mathcal{C}_L)$.*

Proof. One uses enhancements as above and the fact that if $(C_i)_{i\in I}$ is a set of generators for $\mathcal{C}$, then $(\Xi(C_i))_{i\in I}$ is a set of generators for $\mathcal{C}_L$, cf. [38, Prop. 4.8]. $\square$

Remark 3.3.8. Let $\mathcal{C}$ be an abelian category. It is an interesting question whether one could actually define the base change category of $\mathrm{D^b}(\mathcal{C})$ (or $K^b(\mathcal{C})$) simply as $Ho((\mathcal{A}_L)^{pretr})$ for an enhancement $\mathcal{A}$ of $\mathrm{D^b}(\mathcal{C})$. Let us investigate the general case: Let $\mathcal{C}$ be an abelian category with enough injectives and consider $\mathcal{T} = \mathrm{D^b}(\mathcal{C})$. Then $Ho((\mathcal{A}_L)^{pretr}) = \widetilde{K}^+(I(\mathcal{C})_L)$, where the latter denotes bounded-below complexes of objects in $I(\mathcal{C})_L$ with finitely many cohomology objects. The proof of the statement $Ho((\mathcal{A}_L)^{pretr}) = \mathrm{D^b}(\mathcal{C}_L)$ boils down to proving the equivalence $\widetilde{K}^+(I(\mathcal{C})_L) \simeq \widetilde{K}^+(I(\mathcal{C}_L))$, where of course $I(\mathcal{C}_L)$ denotes the category of injective objects in $\mathcal{C}_L$ and hence $\mathrm{D^b}(\mathcal{C}_L) \simeq \widetilde{K}^+(I(\mathcal{C}_L))$. As above it is easy to see that $\widetilde{K}^+(I(\mathcal{C}_L))$ is a full triangulated subcategory in $\widetilde{K}^+(I(\mathcal{C})_L)$. In order to prove that the embedding is essentially surjective, one would in particular have to show that for any injective object $I \in I(\mathcal{C})$ and any module structure f the object (I, f) is in the image. This reduces to the statement that (I, f) is isomorphic to an injective object in $\mathcal{C}_L$. Thus, one has to show the equality $I(\mathcal{C})_L \simeq I(\mathcal{C}_L)$. It is unclear under which conditions this can be proved.

Remark 3.3.9. There are several notions and constructions in the theory of triangulated categories whose preservation under scalar extensions can (and should) be investigated. For example, one could consider the case of quotients: Let $\mathcal{T}$ be a K-linear triangulated category and let $\mathcal{T}' \subset \mathcal{T}$ be a thick triangulated subcategory (recall that this means that $\mathcal{T}'$ is closed under direct summands in $\mathcal{T}$). Then there exists a triangulated category $\mathcal{T}/\mathcal{T}'$, which is the *Verdier quotient* of $\mathcal{T}$ by $\mathcal{T}'$ (for the definition and properties see e.g. [49, Ch. 2]). Let us consider a simple example: Take an abelian category with enough injectives

$\mathcal{C}$, consider its homotopy category $\mathcal{T} = K^b(\mathcal{C})$ and $\mathcal{T}'$, the thick subcategory of acyclic complexes. The quotient is then the derived category of $\mathcal{C}$. Using enhancements as above one can easily see that

$$\mathcal{T}_L/\mathcal{T}'_L \simeq K^b(\mathcal{C}_L)/\mathcal{T}'_L \simeq (\mathcal{T}/\mathcal{T}')_L \simeq \mathrm{D^b}(\mathcal{C}_L).$$

An obvious question is whether the scalar extension construction is always compatible with taking quotients.

Another interesting question is the following: Let $\mathcal{T}$ be a K-linear triangulated category endowed with a Serre functor S. Does the base change category $\mathcal{T}_L$ then also have a Serre functor? The statement holds in the geometric situation but it is unclear whether it can be proved in general.

We conclude this section by sketching a different approach towards the definition of base change which was suggested to us by Prof. V. A. Lunts. Following Keller one calls a triangulated category $\mathcal{T}$ *algebraic* if it is the stable category of a Frobenius exact category (for the definition of the latter see [21, Ch. IV.3, Ex. 4-8]). There is a close connection between algebraic triangulated categories and derived categories of DG-categories (see [37]). We illustrate it in a special case: Namely, by a result of Rouquier [62] the derived category $\mathrm{D^b}(X)$ of a quasi-projective scheme X over a perfect field K is equivalent to $\mathrm{Perf}(A)$, the category of perfect complexes over a DG-algebra A, i.e. the smallest thick subcategory of the derived category of A containing A. Here, A is determined by a strong generator (see Definition 3.4.1 in the next section) E of $\mathrm{D^b}(X)$. To be more precise, $A = \mathrm{RHom}(E, E)$. One could simply define the base change category $\mathrm{D^b}(X)_L$ as $\mathrm{Perf}(A \otimes_K L)$. Let us check that this gives the wanted result: We can assume E to consist of injective objects and therefore $\mathrm{RHom}(E, E)$ is just the complex $\mathrm{Hom}_{DG}(E, E)$. Then by [10, Thm. 2.1.2 and Lem. 3.4.1] the object $p^*(E)$ is a generator of $\mathrm{D^b}(X_L)$ (here we tacitly assume that the field extension is finite since we need $\mathrm{Spec}(L)$ to be a scheme of finite type over $\mathrm{Spec}(K)$). Hence $\mathrm{D^b}(X_L) \simeq \mathrm{Perf}(B)$, where $B = \mathrm{RHom}(p^*(E), p^*(E)) = \mathrm{Hom}_{DG}(p^*(E), p^*(E))$, where the second equality holds because the pullback of an injective sheaf is injective. But

$$B = \mathrm{Hom}_{DG}(p^*(E), p^*(E)) = \mathrm{Hom}_{DG}(E, E) \otimes_K L = A \otimes L.$$

Hence $\mathrm{D^b}(X_L) \simeq \mathrm{Perf}(A \otimes_K L)$.

If one chooses a different generator E', the same proof shows that $\mathrm{Perf}(A' \otimes_K L)$ is again equivalent to $\mathrm{D^b}(X_L)$. Here, the choice of an enhancement is somewhat hidden, but it is indeed present, because we need the DG-structure to define the DG-algebra A.

This definition is certainly more elegant and one could apply it to a vast class of examples, since most triangulated categories arising in algebraic geometry (and representation theory) are in fact algebraic. In the general case one does not have an equivalence between $\mathcal{T}$ and the category of perfect complexes over some DG-algebra, but $\mathcal{T}$ is rather equivalent to (a full subcategory of) the derived category $\mathrm{D}(\mathcal{A})$ of some DG-category $\mathcal{A}$ (for the definition of $\mathrm{D}(\mathcal{A})$ see [36]). For the last statement one has to impose some conditions on $\mathcal{T}$. One could then define the base change category as (a certain subcategory of) the derived category of $\mathcal{A} \otimes_K 1_L$. The disadvantage of this approach is that the DG-algebras resp. DG-categories that appear are in general very difficult to describe.

3.4 Dimension under scalar extensions

In [62] the dimension of a triangulated category was introduced. To recall the definition we need some notation. If $\mathcal{I}$ is a subcategory of a triangulated category $\mathcal{T}$, then $\langle\mathcal{I}\rangle$ denotes the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{I}$ and is closed under isomorphisms, finite direct sums, direct summands and shifts. If $\mathcal{I}_1$ and $\mathcal{I}_2$ are two subcategories, then $\mathcal{I}_1 * \mathcal{I}_2$ is the full subcategory of objects M in $\mathcal{T}$ such that there exists a triangle $I_1 \longrightarrow M \longrightarrow I_2$ with $I_i \in \mathcal{I}_i$. We also define $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle\mathcal{I}_1 * \mathcal{I}_2\rangle$ and set inductively $\langle\,\mathcal{I}\rangle_k = \langle\mathcal{I}\rangle_{k-1} \diamond \langle\mathcal{I}\rangle$. If $\mathcal{I}$ consists of one object E we denote $\langle\mathcal{I}\rangle$ by $\langle E\rangle_1$ and set $\langle E\rangle_k = \langle E\rangle_{k-1} \diamond \langle E\rangle_1$.

Definition 3.4.1. An object E of $\mathcal{T}$ is called a *strong generator* if $\langle E\rangle_n = \mathcal{T}$ for some $n \in \mathbb{N}$. The *dimension* of $\mathcal{T}$ is the smallest integer d such that there exists an object E with $\langle E\rangle_{d+1} = \mathcal{T}$. The *dimension spectrum* of $\mathcal{T}$ is the set of all integers k such that there exists an E with the property that $\langle E\rangle_{k+1} = \mathcal{T}$ but $\langle E\rangle_k \neq \mathcal{T}$.

Example 3.4.2. For a smooth affine scheme Rouquier showed that $\dim(X) = \dim(\mathrm{D}^{\mathrm{b}}(X))$ (cf. [62]). In [55] Orlov showed that the dimension of the bounded derived category of a smooth projective curve C of genus $g \geq 1$ is 1 and conjectured that for any smooth quasi-projective variety X the equality $\dim(\mathrm{D}^{\mathrm{b}}(X)) = \dim(X)$ holds. In the preprint [2] the conjecture was verified for triangulated categories possessing a so called tilting object, which is true e.g. for the derived categories of del Pezzo surfaces with Picard rank at most seven and Hirzebruch surfaces.

One can now ask the following natural

Question: How does the dimension of a triangulated category behave under scalar extensions?

Lemma 3.4.3. *Let L/K be a finite Galois extension with Galois group G, let $\mathcal{C}$ be a K-linear abelian category and $\mathcal{C}_L$ the base change category defined in Section 3.1. Then G acts on $\mathcal{C}_L$ and Galois descent holds, i.e. one has an equivalence between $\mathcal{C}$ and the category $(\mathcal{C}_L)^G$ of objects with Galois-action in $\mathcal{C}_L$ together with Galois-equivariant morphisms.*

Proof. By the Mitchell embedding theorem [41] there exists a full exact embedding of $\mathcal{C}$ into the abelian category $\mathrm{Mod}(A)$ of modules over some K-algebra A. By Example 3.1.6 the category $\mathcal{C}_L$ can then be embedded into $\mathrm{Mod}(A \otimes L)$. It is classical that the pair $\mathrm{Mod}(A)$ and $\mathrm{Mod}(A \otimes L)$ satisfies Galois descent. Let (M, f) be an object with Galois-action in $\mathcal{C}_L$. By Galois descent there exists an A-module N such that $N \otimes_A L$ is isomorphic to (M, f). Considering these modules as modules over A gives an isomorphism $M \simeq N^d$. Since the embedding of $\mathcal{C}$ into $\mathrm{Mod}(A)$ is full and exact, this implies that N is indeed an element in $\mathcal{C}$ (e.g. because it can be written as a kernel of an endomorphism of M). Hence, the pair $\mathcal{C}$ and $\mathcal{C}_L$ fulfils Galois descent as claimed. □

Proposition 3.4.4. *Let $\mathcal{C}$ be an abelian category with enough injectives and with generators and let L/K be a finite Galois extension. Assume that the dimension of* $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$ *is finite. Then* $\dim(\mathrm{D}^{\mathrm{b}}(\mathcal{C})_L) = \dim(\mathrm{D}^{\mathrm{b}}(\mathcal{C}))$.

Proof. We know that $\mathrm{D^b}(\mathcal{C})_L \simeq \mathrm{D^b}(\mathcal{C}_L)$ (Proposition 3.3.7). By the above lemma the category $\mathrm{D^b}(\mathcal{C})$ is dense in $\mathrm{D^b}(\mathcal{C}_L)$, since for any object $A \in \mathrm{D^b}(\mathcal{C}_L)$ the object $\oplus_{g\in G} g^*(A)$ is invariant under the Galois action and hence is isomorphic to an object of $\mathrm{D^b}(\mathcal{C})$. Denote the functor Ξ of Section 3.1 by p^*. If $\langle E\rangle_n^{\mathrm{D^b}(\mathcal{C})} = \mathrm{D^b}(\mathcal{C})$ for some E in $\mathrm{D^b}(\mathcal{C})$, then, by the above argument, $\langle p^*(E)\rangle_n^{\mathrm{D^b}(\mathcal{C}_L)} = \mathrm{D^b}(\mathcal{C}_L)$. This gives the inequality "$\leq$".

For the converse consider a strong generator F in $\mathrm{D^b}(\mathcal{C}_L)$ and denote the dimension of $\mathrm{D^b}(\mathcal{C}_L)$ by n. Assume that an object $M \in \mathrm{D^b}(\mathcal{C})$ can be reached from F in one step, i.e. that there exists a triangle

$$F[-1] \longrightarrow F \longrightarrow p^*(M) \longrightarrow F.$$

Applying all $g \in G$ to this triangle, taking the direct sum and denoting the object $\oplus_{g\in G} g^*(F)$ by $\widetilde{E}$ gives the triangle

$$\widetilde{E}[-1] \longrightarrow \widetilde{E} \longrightarrow \oplus_{g\in G} g^*(p^*(M)) = p^*(M)^{\oplus d} \longrightarrow \widetilde{E}.$$

By Galois descent this is a triangle in $\mathrm{D^b}(\mathcal{C})$ and, therefore, the object $M^{\oplus d}$ can be built from E, where $p^*(E) = \widetilde{E}$, in one step. Induction on the number of steps gives the inequality "$\geq$". □

Proposition 3.4.5. *Let X be a noetherian scheme over K and L/K be a finite Galois extension. If $\dim(\mathrm{D^b}(X))$ is finite, then $\dim(\mathrm{D^b}(X)) = \dim(\mathrm{D^b}(X_L))$. In particular, if $\dim(\mathrm{D^b}(X)) = \dim(X)$, then $\dim(\mathrm{D^b}(X_L)) = \dim(X_L) = \dim(X)$ for any finite Galois extension.*

Proof. Repeat the proof of the previous proposition. □

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